MAT 552. HOMEWORK 7
SPRING 2014
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1. Let G be a closed subgroup of the unitary group $\mathrm{U}(n)$. Use Stone-Weierstrass theorem to show that any continuous function $f \in C(G)$ can be uniformly approximated by linear combinations of matrix coefficients of G-representations.

Definition 1. the Grothendieck group $K(M)$ of a commutative monoid M is a quotient of the set $M \times M$ by the equivalence relation

$$(m_1, n_1) \sim (m_2, n_2) \text{ if } \exists k \in M \text{ such that } m_1 + n_2 + k = m_2 + n_1 + k$$

Addition is defined coordinatewise:

$$(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$$

It is customary to denote element (m_1, n_1) by $[m_1] - [n_1]$.

2.

(1) Prove that equivalence relation is compatible with the multiplicative structure.
(2) Show that $K(M)$ is a group.
(3) Show that the map $m \to [m] - [e]$ defines a homomorphism of monoids
(4) Compute $K(M)$ for $M = \mathbb{Z}_{\geq 0}$
(5) Let X be a finite set and $M = P(X)$ be a monoid of subsets with union $\cup$ being the operation. Compute $K(P(X))$.

Definition 2. A strict monoidal category $< B, \bigcirc, e >$ is a category B with a bifunctor $\bigcirc : B \times B \to B$ which is associative,

$$\bigcirc(\bigcirc \times \mathrm{id}) = \bigcirc(\mathrm{id} \times \bigcirc) : B \times B \times B \to B \tag{1}$$

and with an object e which is a left and right unit for $\bigcirc$

$$\bigcirc(e \times \mathrm{id}) = \mathrm{id}_B = \bigcirc(\mathrm{id} \times e). \tag{2}$$

In writing the associative law (1), we have identified $(B \times B) \times B$ with $B \times (B \times B)$; in writing the unit law (2), we mean $e \times \mathrm{id}$ to be the functor $c \to < e, c >: B \to B \times B$. The bifunctor $\bigcirc$ assigns to each pair of objects $a, b \in B$ an object $a \bigcirc b$ of B and to each pair of arrows $f : a \to a', g : b \to b'$ an arrow $f \bigcirc g : a \bigcirc b \to a' \bigcirc b'$. Thus $\bigcirc$ a bifunctor means that the interchange law

$$\mathrm{id}_a \bigcirc \mathrm{id}_b = \mathrm{id}_{a \bigcirc b}, (f' \bigcirc g')(f \bigcirc g) = (f'f) \bigcirc (g'g),$$

holds whenever the composites $f'f$ and $g'g$ are defined. The associative law (1) states that the binary operation $\bigcirc$ is associative both for objects and for arrows; similarly, the unit law (2) means that $e \bigcirc c = c = c \bigcirc e$ for objects c and that $\mathrm{id}_e \bigcirc f = f = f \bigcirc \mathrm{id}_e$ for arrows f.

A monoidal category B is said to be symmetric when it is equipped with isomorphisms

$$\gamma_{a,b} : a \bigcirc b \cong b \bigcirc a$$

natural in $a, b \in B$, such that the diagrams $\gamma_{a,b}\gamma_{b,a} = \mathrm{id}$, $\mathrm{id}_b = \gamma_{b,e} : b \bigcirc e \cong b$

$$\begin{array}{ccccc} a \bigcirc (b \bigcirc c) & \longrightarrow & (a \bigcirc b) \bigcirc c & \xrightarrow{\gamma} & c \bigcirc (a \bigcirc b) \\ \downarrow \mathrm{id} \bigcirc \gamma & & & & \downarrow \\ a \bigcirc (c \bigcirc b) & \longrightarrow & (a \bigcirc c) \bigcirc b & \xrightarrow{\gamma \bigcirc \mathrm{id}} & (c \bigcirc a) \bigcirc b \end{array}$$

all commute.

Definition 3. Let $< B, \bigcirc, e >$ be a strict symmetric monoidal category. Set of isomorphism classes of objects $M(B)$ is an abelian monoid. The group $K(B) = K(M)$ is the K-group of the category B. Elements of $K(B)$ are called "virtual objects" of B.

3. Let $Rep_k(G)$ be a category of finite-dimensional representations of a compact (Hausdorff) group G over a field k. It is a strict symmetric monoidal category with $\bigcirc = \oplus$.

(1) Show that there is an isomorphism $K_k(G) \cong K(Rep_k(G))$.
(2) A short exact sequence of representations

$$0 \to V \to W \to V' \to 0$$

is an exact sequence vector spaces, whose maps commute with G-action. We define $K'_k(Rep_k(G))$ as a quotient $K(Rep_k(G))$ by relations $[W] = [V] + [V']$. Show that the groups $K_k(Rep_k(G))$ and $K'_k(Rep_k(G))$ coincide.

4. A complex

$$0 \to V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} V_n \to 0$$

of finite-dimensional representations of a compact group G has cohomology groups $H_1, \ldots, H_n$ which are automatically G-representations. Show that a virtual representation $\sum_{i=1}^{n} (-1)^i [V_i]$ is equal to $\sum_{i=1}^{n} (-1)^i [H_i]$