MAT 552. HOMEWORK 6
SPRING 2014
DUE TU MAR 4

We are going to use notations and definitions of HW 5.

Definition 1. Let M be a subset of a topological group G, and $f(x)$ a real valued function defined on . The function $f(x)$ is called uniformly continuous if $\forall\epsilon > 0 \exists$ a neighborhood V of the identity such that $|f(x) - f(y)| < \epsilon$ for $xy^{-1} \in V, x \in M$, and $y \in M$.

Obviously, a uniformly continuous function is continuous.

1. Show that if G a topological (second countable) group, and M a compact subset of G. The continuous function $f(x)$ defined on M is automatically uniformly continuous.(Hint:prove for a circle first)

Definition 2. *A topological space R is called regular if for every neighborhood U of an arbitrary point a there exists a neighborhood V of the same point such that $\overline{V} \subset U$.*

2. Show that the topological space of a topological group G is regular. (Hint: verify this for a neighborhood of $0 \in \mathbb{R}$ and then generalize to a general group)

3. Let G be a compact topological group. Use Urysohn's Lemma to show that for any open set $U \subset G$ there is a nonconstant function $f \in C(G)$ such that $f(x) = 0$ for $x \in G \backslash U$ and $f(x) \geq 0, x \in U$.

Remark 1. *In the course of topology it was proven that a compact regular topological space R satisfying the second axiom of countability is metrizable. Thus any compact topological group is metrizable.*

4. We assume that the group G is compact.

(1) Use the quantity
$$M'(B, f(x)) = \sum_{i=1}^{n} \frac{f(b_i x)}{n}$$
to define a left mean. Denote by G^{op} the group G with new multiplication $x*y = yx$. Verify
$$M(A, M'(B, f(x))) = M'(B, M(A, f(x))). \tag{1}$$
Show that for a continuous function $f \in C(G)$ a right G^{op}-mean coincides with a left G-mean.

(2) Use equation (1) to show that for every $f \in C(G)$ there exists only one right mean and one left mean and these means coincide. The unique mean thus obtained is called the mean of the function f and is denoted by $M(f)$.

(3) Show that $M(M(A, f(x))) = M(f)$.(Hint:use (1) and uniqueness of $M(f)$)
(4) Show that $M(f) + M(g) = M(f + g)$(Hint: use the previous result).
(5) Show that $M(f(xa)) = M(f(x)) = M(f(ax)) \forall a \in G$
(6) If $f(x)$ is a non-negative continuous function defined on G which is not identically zero, then $M(f(x)) > 0$.(Hint: pick a neighborhood U where $f(x) > h > 0$ and find elements $\{a_1, \ldots, a_n\}$ such that $G = \bigcup_{i=1}^{n} Ua_i$)

5. In this problem we establish uniqueness of the integral that satisfies properties (1-5,7) Definition 1 HW5. In the previous problem you verified that $M(f)$ satisfies these conditions. Denote some integral that satisfies these properties by $\int_G^* f \mathrm{d}\mu$

(1) Apply $\int^*$ to $|M(A, f(x)) - p| < \epsilon$ to verify that $\int_G^* f \mathrm{d}\mu = M(f)$. Thus (1-5,7) completely characterize $M(f)$ and we can set $M(f) = \int_G f \mathrm{d}\mu$
(2) Use the previous problem to verify that $M(f(x^{-1})) = M(f(x))$.