MAT 552. HOMEWORK 5
SPRING 2014
DUE TH FEB 27

Definition 1.

We say that an invariant integration is defined over a compact topological group G if the following conditions are satisfied.

(1) To every real continuous function $f(x)$ defined on G corresponds a real number, which we designate symbolically by

$$\int_G f(x)\mathrm{d}\mu,$$

and call the integral of the function $f(x)$ over the group G.

(2) If α, β are real numbers, then

$$\int_G \alpha f(x) + \beta g(x)\mathrm{d}\mu = \alpha \int_G f(x)\mathrm{d}\mu + \beta \int_G g(x)\mathrm{d}\mu$$

,

(3) If $f(x)$ is always non-negative, then $\int_G f(x)\mathrm{d}\mu \geq 0$.

(4) If $f(x) = 1$ for every x, then $\int_G f(x)\mathrm{d}\mu = 1$.

(5) If the function $f(x)$ is non-negative and is not identically zero, then $\int_G f(x)\mathrm{d}\mu > 0$.

(6) If a is an element of G, then $\int_G f(xa)\mathrm{d}\mu = \int_G f(x)\mathrm{d}\mu$.

(7) If a is an element of G, then $\int_G f(ax)\mathrm{d}\mu = \int_G f(x)\mathrm{d}\mu$.

(8) $\int_G f(x^{-1})\mathrm{d}\mu = \int_G f(x)\mathrm{d}\mu$

1. Show that if f, g are continuous functions and $f(x) \leq g(x)$, then

(1) $\int_G f(x)\mathrm{d}\mu \leq \int_G g(x)\mathrm{d}\mu$

(2) $\left|\int_G f(x)\mathrm{d}\mu\right| \leq \int_G |f(x)|\mathrm{d}\mu$

Construction of invariant integration on a compact group G

Definition 2. Let G be a topological group, $f(x)$ a continuous function defined on G, and $A = \{a_1, \dots, a_m\}$ a finite system of elements of the group G. We shall introduce the following notation:

$$M(A, f)(x) = \sum_{i=1}^{m} \frac{f(xa_i)}{m}$$

The function $M(A, f)$ is obviously continuous.

Definition 3. *Let $g(x)$ be a continuous function on a compact space K. We define a variation $Var_G f$ as a difference* $\max_{x\in G} f - \min_{x\in G} f$.

2. Show that

(1) $\max_{x\in G} M(A,f)(x) \le \max_{x\in G} f(x)$
(2) $\min_{x\in G} M(A,f)(x) \ge \min_{x\in G} f(x)$
(3) $Var_G M(A,f)(x) \le Var_G f$
(4) $M(A, M(B,f)) = M(AB,f)$.

3. Show that If $f(x)$ is a non-constant continuous function defined on a compact group G, then there exists in G a finite system A of elements such that

$$Var(M(A,f)) < Var(f) \tag{1}$$

Definition 4.

Let $f(x)$ be a continuous function defined on the compact group G. We shall call a *right mean* of the function $f(x)$ any real number p which possesses the following property: For every positive ϵ there exists a finite system A of elements of the group G such that

$$|M(A, f(x)) - p| < \epsilon \tag{2}$$

4.

(1) Fix a continuous function f on a compact group G. Show that for the family of functions $\Delta = \{M(A,f) | A \subset G, \#A < \infty\}$ conditions of ArzelAscoli theorem are satisfied.
(2) Use compactness of Δ to show that there is a continuous function g such that $Var g \le Var M(A,f)$ for any finite A.
(3) Use the last item of Problem 2 and Problem 3 to show that g is constant.
(4) Show that a continuous function $f(x)$ defined on a compact G has at least one right mean.