1. Suppose \( X \) is a topological space, \( R \) is equivalence relation. Show that
   (1) if the quotient space \( X/R \) is Hausdorff, then \( R \) is closed in the product space \( X \times X \).
   (2) if the projection \( p \) of a space \( X \) onto the quotient space \( X/R \) is open, and \( R \) is closed in \( X \times X \), then \( X/R \) is a Hausdorff space.
   (3) Give an example of equivalence relation \( R \) on the set \( X \) such that \( X \to X/R \) is not open.

**Definition 1.**
(1) A continuous function \( f : X \to Y \) is called proper if \( f \) maps closed sets to closed sets and \( f^{-1}(K) \) is compact for all compact \( K \subset Y \).
(2) Let \( G \) be a topological group acting continuously on a topological space \( X \). The action is called proper if the map \( : G \times X \to X \times X \) given by \((g, x) \to (x, gx)\) is proper.

2. Show that
   (1) If \( G \) acts by homeomorphisms, then the quotient map \( p : X \to X/G \) is always open (contrary to general quotient maps). This is a generalization of Problem 2 HW3.
   (2) \( X/G \) is Hausdorff if and only if the orbit equivalence relation is a closed subset of \( X \times X \).
   (3) If \( G \) acts properly on \( X \) then \( X/G \) is Hausdorff. In particular, each orbit \( Gx \) is closed. The stabilizer \( G_x \) of each point is compact and the map \( G/G_x \to Gx \) is a homeomorphism.
   (4) If \( H \) is a closed subgroup then \( G/H \) is Hausdorff.
   (5) Let \( G \) be a topological group and \( N \) the component of the identity in \( G \). Then \( G/N \) is Hausdorff.

3. (1) Let \( V \) be an inner product space with signature \((1, -1, \ldots, -1)\). Show that if \((l_1, l_1) > 0, (l_2, l_2) > 0 \) then \((l_1, l_2)^2 \geq (l_1, l_1)(l_2, l_2)\)
   (2) Let \( \mathbb{C}^2 \) be a two-dimensional complex space with a basis \( \{e, e'\} \). The space \( \mathbb{C}^2 \otimes \mathbb{C}^2 \)
   has a real structure \( j, j^2 = 1 \) defined by the formula \( j(e \otimes e') = e' \otimes e, j(e \otimes e) = e \otimes e, j(e' \otimes e') = e' \otimes e' \). Identify the space of real points of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) with the space of Hermitian matrices \( M \). Compute the signature of the bilinear form \( \langle A, B \rangle \) associated with the homogeneous quadratic function \( \det A \) on \( M \). Verify that the group \( G_{\mathbb{C}} = \{g \in \text{GL}(2, \mathbb{C})| |\det g| = 1\} \) acts on the space \( M \) by the formula
\( gA = gAg^t \). Compute the signature of \( \langle A, B \rangle \). Identify the group \( Aut(\langle ., . \rangle) \).
Compute the image and the kernel of the homomorphism.

(3) Let \( C^2 \) be a two-dimensional complex space with a basis \( \{e, e'\} \). The space \( C^2 \otimes C^2 \)
has a real structure \( j, j^2 = 1 \) defined by the formula
\[
\begin{align*}
j(e \otimes e') &= e \otimes e', \\
j(e' \otimes e) &= e' \otimes e, \\
j(e \otimes e) &= e \otimes e', \\
j(e' \otimes e') &= e' \otimes e'.
\end{align*}
\]
\( j \) defines a real structure on the symmetric part \( \text{Sym}^2 C^2 \) of \( C^2 \otimes C^2 \).
Let \( M \) be the space of real points in \( \text{Sym}^2 C^2 \).
Compute the signature of the bilinear form \( \langle A, B \rangle \) associated with the homogeneous quadratic function \( \det A \) on \( M \).
Verify that the group \( G_R = \{ g \in \text{GL}(2, \mathbb{R}) | \det g = \pm 1 \} \) acts on the space \( M \) by the formula \( gA = gAg^t \) and preserves \( \langle A, B \rangle \).
Identify the group \( Aut(\langle ., . \rangle) \) and compute the image of the homomorphism \( G_R \rightarrow Aut(\langle ., . \rangle) \).

(4) One dimensional quaternionic space \( \mathbb{H} \) is the same as two-dimensional complex space \( C^2 \) with a structure map \( j, j^2 = -1 \).
The space \( C^4 = C^2 + C^2 \) carries the diagonal structure map \( j \).
Let \( \Lambda^2 C^4 \) be the skew-symmetric part of \( C^4 \otimes C^4 \).
\( j \otimes j \) defines a real structure \( (j^2 \otimes j^2 = \text{id} \otimes \text{id}) \) on \( \Lambda^2 C^4 \).
The Pfaffian function \( \text{Pf}(A) \) can be used to define a bilinear form \( \langle A, B \rangle \) on \( \Lambda^2 C^4 \).
Compute the signature of \( \langle A, B \rangle \).
Verify that the group \( G_H = \{ g \in \text{GL}(4, \mathbb{C}) | \det g = \pm 1, gj = jg \} \) acts on the space \( M \) by the formula \( gA = gAg^t \) and preserves \( \langle A, B \rangle \).
Compute the image of the homomorphism \( G_H \rightarrow Aut(\langle ., . \rangle) \).