MAT 552. HOMEWORK 2
SPRING 2014
DUE TU FEB 11

1. The group $Aut(\mathbb{R}^n) = \mathrm{GL}(n)$ is an open subset in Mat_n and admits a single chart. In this chart the tangent bundle $T(\mathrm{GL}(n))$ is a product $\mathrm{GL}(n) \times Mat_n$. Write an explicit formula for a left-invariant vector field.

2. Smooth maps from a manifold M to $\mathbb{R}$ form an algebra $C^\infty(M)$. A derivation $D : C^\infty(M) \to C^\infty(M)$ is a linear map such that for any smooth real function $g(x_1, \ldots, x_n)$ and n-tuple $s_1, \ldots, s_n \in C^\infty(M)$

$$Dg(s_1, \ldots, s_n) = \sum_{i=1}^{n} \frac{\partial g(s_1, \ldots, s_n)}{\partial x_i} Df_i \tag{1}$$

Assuming if necessary that M is paracompact show that

(1) D satisfies Leibniz rule.
(2) Show that if f_1 and $f_2 \in C^\infty(M)$ coincide in some neighborhood of $m \in M$ then so do $D(f_1)$ and $D(f_2)$.
(3) There is a one-to-one correspondence between derivations of $C^\infty(M)$ and sections of $T(M)$.
(4) A derivation at a point $m \in M$ is a linear map $D_m : C^\infty(M) \to \mathbb{R}$ that satisfies $D_m g(s_1, \ldots, s_n) = \sum_{i=1}^n \frac{\partial g(s_1,\ldots,s_n)}{\partial x_i}(m) D_m s_i$. Show that there is a natural identification of set of such D_m with the tangent space T_m.

Warning!: local coordinates are not global functions on M and cannot be differentiated by D!

Definition 1. (1) *A Lie algebra* $\mathfrak{g}$ *is a real vector space equipped with a bilinear map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, *which is skew-symmetric*

$$[a, b] = -[b, a]$$

and satisfies Jacobi identity

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$$

(2) *A homomorphism of Lie algebras* $\phi : \mathfrak{g} \to \mathfrak{h}$ *is a linear map that satisfies* $\phi[a, b] = [\phi a, \phi b]$.
(3) *An automorphism is an invertible homomorphism from Lie algebra to itself.*
(4) *A linear subspace* $\mathfrak{h} \subset \mathfrak{g}$ *is a subalgebra if it is stable under commutator.*

3.

(1) Using results of the previous problem show that vector fields (sections of a tangent bundle) interpreted as derivations form a Lie algebra $Vect(M)$ with a commutator $[D_1, D_2] = D_1 D_2 - D_2 D_1$

(2) Show that any smooth homeomorphism (diffeomorphism) of a manifold defines an automorphism of the algebra $C^\infty(M)$ and the Lie algebra $Vect(M)$

(3) A vector field $\lambda \in Vect(M)$ is invariant with respect to diffeomorphism $f \in Diff(M)$. if $f_* \lambda = \lambda$. Let G be a subgroup of $Diff(M)$. Show that $Vect(M)^G = \{\lambda \in Vect(M) | g_* \lambda = \lambda\}$ is a Lie subalgebra.

(4) The space of left-invariant vector fields on a Lie group G has a structure of a Lie algebra.