MAT 552. HOMEWORK 10
SPRING 2014
DUE TU APR 29

- Denote by $[A, B]$ the commutator of two operators.
- Let $C^\infty(M)$ be a linear space of smooth functions on a manifold M. Multiplication on a smooth function f defines an operator $f : C^\infty(M) \to C^\infty(M)$. A linear map $L : C^\infty(M) \to C^\infty(M)$ is a differential operator if there is $k > 0$ such that $[f_k, \dots, [f_1, L] \dots] = 0$ for any k-tuple of smooth functions $(f_1, \dots, f_k)$.
- The order (or degree) $ord(L)$ of a differential operator L is number $k - 1$ from the previous definition.
- Denote the set of differential operators of degree n by $\mathrm{Diff}_n(M)$.

1.

(1) Show that if $f \in C^\infty(M)$ and $L \in \mathrm{Diff}_k(M)$ then $[f, L] \in \mathrm{Diff}_{k-1}(M)$.
(2) Show that $\mathrm{Diff}_i(M)\mathrm{Diff}_j(M) \subset \mathrm{Diff}_{i+j}(M)$.
(3) Show that $A \in \mathrm{Diff}_i(M), B \in \mathrm{Diff}_j(M) \Rightarrow [A, B] \in \mathrm{Diff}_{i+j-1}(M)$.
(4) Identify $\mathrm{Diff}_0(M)$.
(5) Identify $\mathrm{Diff}_1(M)$.

- Let $T(M)$ be the tangent bundle. $\mathrm{Sym}^k T(M)$ is k-th symmetric power of the tangent bundle. If M is modeled on $\mathbb{R}^n$, then the gluing maps of the tangent bundle are $\psi'_{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}^n$. The gluing maps of $\mathrm{Sym}^k T(M)$ are $\psi'^{\otimes k}_{\alpha\beta} : \mathrm{Sym}^k \mathbb{R}^n \to \mathrm{Sym}^k \mathbb{R}^n$.

2.

(1) Let $L \in \mathrm{Diff}_k(M)$ and $J \subset C^\infty(M)$ is an ideal of function vanishing at $m \in M$. Show that $L(J^{k+1}) \subset J$.
(2) Generalized this result on powers of an arbitrary ideal.
(3) Show that differential operators are localizable, which means that restriction on an open chart $U \subset M$ is well defined. Moreover a differential operators preserve functions that vanish outside some open set U such that $V = M \backslash U$ satisfies $V = \overline{\mathring{V}}$.
(4) Let $L \in \mathrm{Diff}_k(M)$ and $x_i, \dots, x_n$ be local coordinates in a neighborhood $U \subset M$. Then operator $L|_U$ can be presented in the form

$$L|_U = \sum_{|\sigma|=0}^{k} \alpha_\sigma(x) \frac{\partial^\sigma}{\partial x^\sigma}, \quad \alpha_\sigma(x) \in C^\infty(U)$$

where $\sigma = (i_1, \dots, i_n)$ is a multiindex , $|\sigma| = \sum_{s=1}^n i_s$
(5) Show that $\mathrm{Diff}_k(M)/\mathrm{Diff}_{k-1}(M)$ is isomorphic to the space of sections of $\mathrm{Sym}^k T(M)$

- Define $U(\mathfrak{g})$ the universal enveloping algebra of a Lie algebra of a Lie group G as a subalgebra $\mathrm{Diff}(G)^G$of algebra of differential operators $\mathrm{Diff}(G) = \bigcup_{k\geq 0} \mathrm{Diff}_k(G)$ that commute with right translations : $L_g(f(gh)) = L_g(f)(gh)$.
- Filtration $\mathrm{Diff}_k(G)$ induces filtration in $\mathrm{Diff}(G)^G = U(\mathfrak{g})$, which we denote by $F_kU(\mathfrak{g})$

3.

(1) Show that there is a an isomorphism $F_kU(\mathfrak{g})/F_{k-1}U(\mathfrak{g}) \cong \mathrm{Sym}^k\mathfrak{g}$ (Poincaré-Birkhoff-Witt theorem).
(2) Show that $F_1U(\mathfrak{g}) \cong \mathbb{R} + \mathfrak{g}$.
(3) Show that $F_1U(\mathfrak{g})$ generates $U(\mathfrak{g})$
(4) Pick a basis $\{l_i | i = 1 \dim \mathfrak{g}\}$, show that

$$l_il_j - l_jl_i = [l_i, l_j] \tag{1}$$

in $U(\mathfrak{g})$
(5) Show that (1) are defining relations in $U(\mathfrak{g})$.