1. (1) Prove that $M$ is Hausdorff space ⇔ the diagonal $\Delta \subset M \times M$ is closed.
(2) Show that a topological group $G$ is Hausdorff ⇔ one point set $\{e\} \subset G$ where $e$ is the unit is closed.
(3) Recall that Lie group is a topological group, which is Hausdorff and which admits an atlas of neighborhoods homeomorphic to $\mathbb{R}^n$ together with smoothness condition. Show that the Hausdorff property is automatically satisfied.

2. Let $S^3$ be a round sphere

$$S^3 = \{x \in \mathbb{R}^4 | x \cdot x = 1\}$$

Identify the space of pairs

$$X = \{(x, v) \in \mathbb{R}^4 \times \mathbb{R}^4 | x \cdot x = 1, x \cdot v = 0\}$$

with the tangent bundle to $S^3$ as it was defined in class.

**Definition 1.** Cayley transform is a map $\text{Mat}_n \rightarrow \text{Mat}_n$

$$A^# = (\text{id} - A)(\text{id} + A)^{-1}$$

defined for all matrices such that $\det(\text{id} + A) \neq 0$. We denote the set of such matrices by $R_n$.

3. (1) Prove that $### = \text{id}$ and $#(R_n) \subset R_n$.
(2) Fix a standard inner product on $\mathbb{R}^n$. Let $O(n) = \{A \in \text{Mat}_n | AA^t = \text{id}\}$. Show that $#(O(n) \cap R_n) = \Lambda^2 \mathbb{R}^n \cap R_n$, where $\Lambda^2 \mathbb{R}^n$ is a linear space of skew-symmetric matrices.
(3) Prove that $O(n)$ is a Lie group.