Problem 1  1. Prove that trigonometric polynomials of the form

\[ a_0 + \sum_{n=0}^{k} a_n \sin(2\pi n x) + b_n \cos(2\pi n x), a_i, b_j \in \mathbb{R} \]

form a dense set is the space \( C(T) \) of continuous functions of one variable that satisfy \( f(x) = f(x + 1) \).

2. Prove that continuous functions on units disk can be uniformly approximated by functions of the form

\[ a_0 + \sum_{n=0}^{k} r^n(a_n \sin(2\pi n \theta) + b_n \cos(2\pi n \theta)), a_i, b_j \in \mathbb{R} \]

where \((r, \theta) \in [0, 1] \times [0, 1]\) are polar coordinates.

3. Let \( I_2 = [0, 1] \times [0, 1] \) be a square. Show that any continuous function on \( I_2 \) can be uniformly approximated by functions having the form:

\[ \sum_{i=1}^{n} f_i(x)g_i(y) \]

where \( f_i, g_i \) are continuous functions on \([0, 1]\).

Problem 2  Prove that

- The space \( \mathbb{R}^2 \) is not a union of a countable set of lines.
- The set of irrational numbers \( \mathbb{R} \backslash \mathbb{Q} \) is not a union of closed subsets, none of which contains a open subset.
- Can a closed set be dense? Give an example.

Problem 3  Suppose that \( \{f_n\} \) is a sequence of continuous functions on \([0, 1]\) and that for every \( x \in [0, 1] \)

\[ \lim_{n \to \infty} f_n(x) = f(x). \]

Prove that \( f(x) \) must be continuous at some point of \([0, 1]\). (Hint for any \( \epsilon > 0 \) apply Baire category to the sets \( A_\epsilon = \{x||f(x) - f_n(x)\| \leq \epsilon\} \) to show that there is an interval \( I \subset [0, 1] \) on which \( \sup_I f(x) - \inf_I f(x) < \epsilon \).)
**Problem 4** Use Tietze theorem to show that if $F$ is closed subset of a metric space $X$ and $f$ is a (possibly unbounded) continuous function on $F$ then there is continuous extension, that is defined on all $X$.

**Problem 5** Find an interval on which there is a solution with given initial conditions:

1. $y' = x + y^3, y(0) = 0.$
2. $y' = x + \exp(y), y(1) = 0$

**Problem 6** Construct the second approximation of the solution of the following equation using the Picard method

$$y' = x - y^2, 0 \leq x \leq 1/2, y(0) = 0$$

and estimate the error of approximation.

**Problem 7** For what values of $\alpha, \beta \in \mathbb{R}$ and in what region one can guarantee

1. local existence
2. local uniqueness

of the solution of the following differential equation

$$x' = |t|^\alpha + |x|^\beta$$

**Problem 8** For what value of $n$ Bernstein polynomial $B_n(x, f)$ approximates function

1. $f(x) = |x - 1/2|$ 
2. $f(x) = x^3$

with an error $\epsilon = 1/10$ on the interval $[0, 1]$. Draw the graph of the approximation as a function $B : \mathbb{R} \to \mathbb{R}$ (you may use graphing calculator Maple or Mathematica as an aid)

Investigate convergence of $B_n(x, f)$ where $f = 0$ if $x < 1/2$ and $f = 1$ if $x \geq 1/2$. What can you say about convergence of $B_n$ outside of the interval $[0, 1]$?