MAT544 Fall 2009

Homework 6

Problem 1 1. Prove that trigonometric polynomials of the form

$$a_0 + \sum_{n=0}^{k} a_n \sin(2\pi nx) + b_n \cos(2\pi nx), a_i, b_j \in \mathbb{R}$$

form a dense set is the space $C(\mathbf{T})$ of continuous functions of one variable that satisfy $f(x) = f(x+1)$.

2. Prove that continuous functions on units disk can be uniformly approximated by functions of the form

$$a_0 + \sum_{n=0}^{k} r^n(a_n \sin(2\pi n\theta) + b_n \cos(2\pi ntheta)), a_i, b_j \in \mathbb{R}$$

where $(r, \theta) \in [0,1] \times [0,1]$ are polar coordinates.

3. Let $I_2 = [0,1] \times [0,1]$ be a square. Show that any continuous function on I_2 can be uniformly approximated by functions having the form:

$$\sum_{i=1}^{n} f_i(x) g_i(y)$$

where f_i, g_i are continuous functions on $[0,1]$.

Problem 2 Prove that

- The space $\mathbb{R}^2$ is not a union of a countable set of lines.
- The set of irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ is not a union of closed subsets, non of which contains a open subset.
- Can a closed set be dense? Give an example.

Problem 3 Suppose that $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ and that for every $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Prove that $f(x)$ must be continuous at some point of $[0,1]$. (Hint for any $\epsilon > 0$ apply Baire category to the sets $A_n = \{x \| f(x) - f_n(x)| \leq \epsilon\}$ to show that there is an interval $I \subset [0,1]$ on which $\sup_I f(x) - \inf_I f(x) < \epsilon$.)

Problem 4 Use Tietze theorem to show that if F is closed subset of a metric space X and f is a (possibly unbounded) continuous function on F then there is continuous extension, that is defined on all X.

Problem 5 Find an interval on which there is a solution with given initial conditions:

1. $y' = x + y^3, y(0) = 0$.

2. $y' = x + \exp(y), y(1) = 0$

Problem 6 Construct the second approximation of the solution of the following equation using the Picard method

$$y' = x - y^2, 0 \le x \le 1/2, y(0) = 0$$

and estimate the error of approximation.

Problem 7 For what values of $\alpha, \beta \in \mathbb{R}$ and in what region one can guarantee

1. local existence

2. local uniquness

of the solution of the following differential equation

$$x' = |t|^\alpha + |x|^\beta$$

Problem 8 For what value of n Bernstein polynomial $B_n(x, f)$ approximates function

1. $f(x) = |x - 1/2|$

2. $f(x) = x^3$

with an error $\epsilon = 1/10$ on the interval $[0, 1]$. Draw the graph of the approximation as a function $B : \mathbb{R} \to \mathbb{R}$ (you may use graphing calculator Maple or Mathematica as an aid)

Investigate convergence of $B_n(x, f)$ where $f = 0$ if $x < 1/2$ and $f = 1$ if $x \ge 1/2$. What can you say about convergence of B_n outside of the interval $[0, 1]$?