MAT544 Fall 2009

Homework 5

Let $Lip[a,b]$ be a subset $\{f\}$ of space of functions on the interval $[a,b]$ which satisfy

$$|f|_{Lip} = \sup_{[a,b]} |f(x)| + \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|} < +\infty$$

Problem 1 Prove that

- Lip is a linear space. The space of infinitely differentiable functions $C^\infty[a,b]$ is a subspace in Lip.
- $C^\infty[a,b]$ is not a dense subset in Lip. Hint:Show that the function

$$f(x) = \begin{cases} 0 \text{ if } a \le x \le \frac{a+b}{2} \\ x - \frac{a+b}{2} \text{ if } \frac{a+b}{2} \le x \le b \end{cases} \tag{1}$$

  can not be approximated by a smooth function $g \in C^\infty[a,b]$

Problem 2 Let f_n be a sequence of differentiable functions on interval $[0,1]$. Assume there are constants c_0, c_1 such that

$$\sup_{[0,1]} |f_n(x)| \le c_0, \qquad \int_0^1 (f'(x))^2 dx \le c_1$$

Prove that $\{f_n(x)\}$ contains a uniformly converging subsequence.

Problem 3 Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ to $\mathbb{R}$ which converges at each point of the set $\mathbb{Q}$ of rationals. If the set $\mathcal{F} = \{f_n\}$ is equicontinuous on $\mathbb{R}$, show that the sequence is actually convergent at every point of $\mathbb{R}$ and that this convergence is uniform on $\mathbb{R}$.

Problem 4 Let $\mathcal{F}$ be a bounded and equicontinuous set of functions with domain M contained in metric space X and with range in $\mathbb{R}$. Let $f^*(x)$ be defined on M to $\mathbb{R}$ by

$$f^*(x) = \sup_{\mathcal{F}} f(x).$$

Show that $f^*(x)$ is continuous on M to $\mathbb{R}$.

Problem 5 Show that a unit sphere in a Hilbert space E is compact iff E is finite-dimensional.

Problem 6 Let M be a subset in $C[a,b]$ that consists of polynomials of degree ≤ 10, that satisfy inequality $\int_a^b |p(x)|dx \leq 10$. Is the space M compact?

Problem 7 Let A be a map from $C[0,1]$ to itself defined by the formula

$$A(f)(x) = \lambda \int_0^1 K(x,y)f(y)dy + g(x),$$

where $K(x,y) \in C([0,1]\times[0,1]), g(x) \in C[0,1]$. Find a sufficient condition that the equation $Af = f$ has a solution.

Problem 8 Construct a function $f : [0,1] \to \mathbb{R}$ or prove that such construction is impossible:

- A function that is discontinuous at all points.
- A function that is discontinuous at all rational points.
- A function that is discontinuous at all irrational points.