MAT544 Fall 2009

Homework 4

Problem 1 Determine $\underline{\lim}_{x\to 0} f(x)$, $\overline{\lim}_{x\to 0} f(x)$

1. $f(x) = \sin^2(1/x) + \frac{2}{\pi}\mathrm{arctg}(1/x)$
2. $f(x) = \left(1 + \cos^2 \frac{1}{x}\right)^{\frac{1}{\cos^2(1/x)}}$

A map $f : X \to Y$ between two metric spaces X, Y is called uniformly continuous if for any $\epsilon > 0$ there is $\delta > 0$ such that $\rho_Y(f(x), f(y)) < \epsilon$ if $\rho_X(x, y) < \delta$.

Problem 2 Prove that the functions

1. $f(x) = 1/x$ is continuous but not uniformly continuous on the interval $(0, 1)$.
2. $f(x) = \sin(\frac{\pi}{x})$ is continuous bounded but not uniformly continuous on the interval $(0, 1)$.
3. $f(x) = x + \sin(x)$ is uniformly continuous on $\mathbb{R}$.

Problem 3

1. Prove that if a bounded continuous function $f(x)$ is nondecreasing on a finite or infinite interval (a, b) then $f(x)$ is uniformly continuous on this interval.
2. Prove that if a function $f : [a, +\infty) \to \mathbb{R}$ is continuous and has a limit
$$\lim_{x\to+\infty} f(x)$$
then $f(x)$ is uniformly continuous on $[a, +\infty)$.
3. Prove that if a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous then there are limits
$$\lim_{x\to a+0} f(x), \lim_{x\to b-0} f(x)$$

The modulus of continuity of a function $f : (a, b) \to \mathbb{R}$ by definition is the number

$$\omega_f(\delta) = \sup_{|x_1-x_2|\le\delta} |f(x_1) - f(x_2)|$$

Problem 4

1. Prove that $f(x)$ is uniformly continuous on (a, b) iff $\lim_{\delta\to 0} \omega_f(\delta) = 0$.
2. Find an estimate for $\omega_f(\delta)$ of the form

$$\omega_f(\delta) \le C\delta^\alpha,$$

where C, α are suitable constants if

(a) $f(x) = x^3$, $(0 \le x \le 1)$

(b) $f(x) = \sin(x) + \cos(x)$, $(0 \le x \le 2\pi)$

Let A be a subset of a metric space X. The diameter $diam(A)$ by definition is $\sup_{x,y\in A} \rho(x, y)$ Let $B_r(x) = \{y \in X|\rho(y, x) < r\}$ be an open ball in X of radius r.

Problem 5 Let E be a normed space and $B_r(x)$ is a ball in E.

1. Show that $\overline{B}_r(x) = \{y \in X|\rho(y, x) \le r\}$
2. Show that $diam(B_r(x)) = 2r$
3. Show that if $B_r(x) \subset B_R(y) \subset E$. Then $r \le R$ and $||x - y|| \le R - r$

A set $A \subset E$ is called bounded if there is $r \ge 0$ such that $A \subset \overline{B}_r(0)$

Problem 6

1. Show that A is bounded iff $diam(A) < \infty$.
2. Show that A is bounded iff for any sequence $x_n \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$ and for any sequence $v_n \in A$ the limit $\lim_{n\to\infty} x_n v_n = 0$.

Let A, B be two subsets in a normed vector space E. Define $A + B = \{a + b|a \in A, b \in B\}$.

Problem 7 Show that is A and B are not empty and A is open then $A + B$ is open.

The distance between two subsets A, B in the metric space X is defined as

$$\inf_{a\in A, b\in B} \rho_X(a, b).$$

Problem 8 Show that $\rho(A, B) = \rho(A, \overline{B})$.