MAT544 Fall 2009

Homework 10

Problem 1 Using results of HW9 prove that

1. All norms $||x||_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$, $p \geq 1$ and $||x||_\infty = \max_{i=1,\ldots,n} |x_i|$ are equivalent on $\mathbb{R}^n$.

2. Define l_p as a space of sequences $x = (x_1, \ldots, x_n, \ldots)$ such that $\sum_{n\geq 1} |x_n|^p < \infty$, $p \geq 1$. Prove that l_p is a Banach space with respect to l_p norm

$$||x||_p = \sqrt[p]{\sum_{n\geq 1} |x_n|^p}$$

.

3. Show that if $1/p + 1/q = 1$ then the pairing $l_p \times l_q \to \mathbb{R}$ defined by the formula

$$(x_1, \ldots, x_n, \ldots) \times (y_1, \ldots, y_n, \ldots) \to \sum_{n\geq 1} x_n y_n$$

is well defined and continuous in both arguments.

Problem 2 Let $E_i, i \geq 1$ be a collection of subsets of a set X. Define

$$\overline{\lim_{i\to\infty}} E_i = \bigcap_{i\geq 1} \bigcup_{n\geq i} E_n$$

and

$$\underline{\lim_{i\to\infty}} E_i = \bigcup_{i\geq 1} \bigcap_{n\geq i} E_n$$

This you probably already know from HW9, but still:

Prove that

1. $\overline{\lim_{i\to\infty}} E_i = \{x \in X | x \text{ contains in infinitely many } E_i\}$

2. $\underline{\lim_{i\to\infty}} E_i = \{x \in X | x \text{ contains in all } E_i \text{ starting with some } i_0\}$

3. $\underline{\lim_{i\to\infty}} E_i \subset \overline{\lim_{i\to\infty}} E_i$

Problem 3 Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called locally measurable if $E \cap A \subset \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called saturated.

a If μ is σ-finite, then μ is saturated. (Recall if $X = \bigcup E_j$ where $E_i \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, μ is called σ-finite.)

b $\widetilde{\mathcal{M}}$ is a σ-algebra.

c Define $\tilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the saturation of μ.

d If μ is complete, so is $\tilde{\mu}$.

e Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) | A \in \mathcal{M}$ and $A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ. (Recall if for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, μ is called semifinite.)

f Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.

Problem 4 Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ be a collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ be a collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* be the induced outer measure on $\mathcal{P}(X)$. Prove that

1. For any $E \subset X$ and $\epsilon > 0$ there is $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) < \mu^*(E) + \epsilon$

2. If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \backslash E) = 0$(Hint: for B take the intersection of sets from the first part of the problem. Use that B is measurable (proven in class)).

Problem 5 Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define an inner measure $\mu_*(E)$ to be $\mu_*(E) = \mu_0(X) - \mu^*(\overline{E})$. Prove that E is measurable iff $\mu^*(E) = \mu_*(E)$. (Hint: use the result of the second part of the previous problem.)

Problem 6 Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq \infty$.

a $\mathcal{A}$ is an algebra on $\mathbb{Q}$.

b The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

c Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Problem 7 Let m denote Lebesgue measure. Let $E \subset [0, 1]$, and let $\overline{E}$ be the closure of E. Show that $m(\overline{E}) = 0$ iff for every $\epsilon > 0$ there is a *finite* disjoint collection of open intervals $I_j \subset \mathbb{R}$ with $E \subset \cup I_j$ and with $\sum m(I_j) < \epsilon$.

Problem 8 A linear map $T : E_1 \to E_2$ between two Banach spaces is called bounded if it satisfies $||T(x)||_2 \leq K||x||_1$, where $K \geq 0$. A trivial example of bounded map is a map proportional to identity map.

a Give a nontrivial example of a bounded map $T : l_2 \to l_2$ with no kernel and find the constant K.

b Find a bounded $T : l_2 \to l_2$ with dense image $\mathrm{Im}T \subset l_2$ such that $\mathrm{Im}T \neq l_2$