1 Semifields

Definition 1 A semifield $\mathbb{P} = (\mathbb{P}, \oplus, \cdot)$:

1. $(\mathbb{P}, \cdot)$ is an abelian (multiplicative) group.
2. $\oplus$ is an auxiliary addition: commutative, associative, multiplication distributes over $\oplus$.

Exercise 1 Show that semi-field $\mathbb{P}$ is torsion-free as a multiplicative group. Why doesn’t your argument prove a similar result about fields?

Exercise 2 Show that if a semi-field contains a neutral element $0$ for additive operation and $0$ is multiplicatively absorbing

$$0a = a0 = 0$$

then this semi-field consists of one element

Exercise 3 Give two examples of non injective homomorphisms of semi-fields

Exercise 4 Explain why a concept of kernel is undefined for homorphisms of semi-fields.

A semi-field $\text{Trop}_{\text{min}}$ as a set coincides with $\mathbb{Z}$. By definition $a \cdot_{\text{Trop}} b = a + b$, $a \oplus b = \min(a, b)$. Similarly we define $\text{Trop}_{\text{max}}$.

Exercise 5 Show that $\text{Trop}_{\text{min}} \cong \text{Trop}_{\text{max}}$

Let $\mathbb{Z}[u_1, \ldots, u_n]_{\geq 0}$ be the set of nonzero polynomials in $u_1, \ldots, u_n$ with non-negative coefficients.

A free semi-field $\mathbb{P}(u_1, \ldots, u_n)$ is by definition a set of equivalence classes of expression $\frac{P}{Q}$, where $P,Q \in \mathbb{Z}[u_1, \ldots, u_n]_{\geq 0}$.

$$\frac{P}{Q} \sim \frac{P'}{Q'}$$

if there is $P'', Q'', a, a'$ such that $P'' = aP = a'P', Q'' = aQ = a'Q'$.
Exercise 6 Show that for any semi-field \( P \) and a collection \( v_1, \ldots, v_n \) there is a homomorphism

\[
\psi : P(u_1, \ldots, u_n) \to P', \psi(u_i) = v_i
\]

Let \( k \) be a ring. Then \( k[P] \) is the group algebra of the multiplicative group of the semi-field \( P \).

2 Cluster algebras - foundations

Definition 2 \( B = (b_{ij}) \) is an \( n \times n \) integer matrix is skew-symmetrizable if there exists a diagonal matrix \( D \) with positive diagonal entries such that \( DBD^{-1} \) is skew-symmetric.

Exercise 7 Show that \( B \) is skew-symmetrizable if and only if there exist positive integers \( d_1, \ldots, d_n \) such that \( d_i b_{ij} = -d_j b_{ji} \) for all \( i \) and \( j \).

Definition 3 An exchange matrix is a skew-symmetrizable \( n \times n \) matrix \( B = (b_{ij}) \) with integer entries.

Let \( F \) be purely transcendental extension (of transcendental degree \( n \)) of the field of fractions \( \mathbb{Q}(P) \) of \( \mathbb{Q}[P] \).

Definition 4 A labeled seed is a triple \((x, y, B)\), where

- \( B \) is an \( n \times n \) exchange matrix,
- \( y = (y_1, \ldots, y_n) \) is a tuple of elements of \( P \) called coefficients, and
- \( x = (x_1, \ldots, x_n) \) is a tuple (or cluster) of algebraically independent (over \( \mathbb{Q}(P) \)) elements of \( F \) called cluster variables.

A pair \((y, B)\) is called a \( Y \)-seed.

Definition 5 Let \( B = (b_{ij}) \) be an exchange matrix. Write \([a]_+ \) for \( \max(a, 0) \).

The mutation of \( B \) in direction \( k \) is the matrix \( b'_{ij} \) where

\[
b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } k \in \{i, j\} \\
-b_{ij} + \text{sign}(b_{kj})[b_{ik}b_{kj}]_+, & \text{otherwise}
\end{cases}
\]
Exercise 8 Show that $\mu_k(B)$ is an exchange matrix, e.g. it is skew-symmetrizable.

Exercise 9 Show that matrix mutation can be equivalently defined by

$$b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } k \in \{i,j\} \\
 b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+, & \text{otherwise}
\end{cases}$$

Definition 6 Let $(y, B)$ be a $Y$-seed. The mutation of $(y, B)$ in direction $k$ is the $Y$-seed $(y', B') = \mu_k(y, B)$, where $B' = \mu_k(B)$ and $y'$ is the tuple $(y'_1, \ldots, y'_n)$ given by

$$y'_j = \begin{cases} 
 y_k^{-1}, & \text{if } j = k \\
 y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}}, & \text{if } j \neq k
\end{cases}$$

Definition 7 Let $(x, y, B)$ be a labeled seed. The mutation of $(x, y, B)$ in direction $k$ is the labeled seed $(x', y', B') = \mu_k(x, y, B)$, where $(y', B')$ is the mutation of $(y, B)$ and where $x'$ is the cluster $(x'_1, \ldots, x'_n)$ with $x'_j = x_j$ for $j \neq k$, and

$$x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{-[b_{ik}]_+}}{(y_k \oplus 1)x_k}$$

Exercise 10 Show that each mutation $\mu_k$ is an involution on labeled seeds.

Applying several mutations $\mu_{i_1} \cdots \mu_{i_l}$ to a labeled seed $(x, y, B)$ we get a new labeled seed. Let $\Delta_n(x, y, B)$ be the set of all such seeds.

Definition 8 A cluster algebra $A(x, y, B)$ is a subalgebra in $\mathbb{Q}(\mathbb{P})(x_1, \ldots, x_n)$ generated by all cluster variables in $\Delta_n(x, y, B)$.

Definition 9 Let $\tilde{B}$ be $(m+n) \times n$ matrix, such that the top $n \times n$ matrix is skew-symmetrizable and $\tilde{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$. Then we say that $(\tilde{x}, \tilde{B})$ is a labeled seed for a cluster algebra of geometric type. Collection $(x_1, \ldots, x_n)$ is known as exchangeable variables; $(x_{n+1}, \ldots, x_{n+m})$ as frozen variables or "coefficients". Notation: $(u_1, \ldots, u_m)$ is occasionally used for frozen variables.

Let $\tilde{x}' = \mu_k(\tilde{x})$, $\tilde{B}' = \mu_k(\tilde{B})$, $k = 1, \ldots, n$. Then $\mu_k(\tilde{B})$ is defined as in $n \times n$ case: $x'_j = x_j$, $j \neq k$

$$x'_k = \frac{\prod x_i^{[b_{ik}]_+} + \prod x_i^{-[b_{ik}]_+}}{x_k}$$

3
Definition 10 Let $\Delta_n(x, B)$ be the set of mutations of geometric seed $(x, B)$. By definition cluster algebra of geometric type as a subalgebra in $\mathbb{Q}(x_1, \ldots, x_{n+m})$ generated by cluster variables in $\Delta_n(x, B)$.

Exercise 11 Let $\mathbb{P}$ a tropical semi-field on $n$ generators $y_1, \ldots, y_n$. Show that the homomorphism of fields $\phi : \mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n) \rightarrow \mathbb{Q}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ identical on $x_1, \ldots, x_n$ and on $y_1, \ldots, y_n$ defined by the formula:

$$\phi(y_j) = \prod_{i=1}^{m} x_{n+i}^{b_{n+i,j}}$$

is compatible with mutations.

Exercise 12 Consider the cluster algebra of geometric type defined by the initial labeled seed given by $x = (x_1, x_2, u_1, u_2, u_3)$ and

$$B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Compute all cluster variables generating this cluster algebra.
3 Root systems

Definition 11 Given a nonzero vector \( \alpha \) in Euclidean space \( V \), the reflection in the hyperplane orthogonal to \( \alpha \) is \( \sigma_\alpha \), given by

\[
\sigma_\alpha(x) = x - 2\left(\frac{\alpha}{\sqrt{\langle \alpha, \alpha \rangle}}, x\right) \cdot \left(\frac{\alpha}{\sqrt{\langle \alpha, \alpha \rangle}}\right) = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}
\]  (1)

Define \( \alpha^\vee = 2\frac{\alpha}{\langle \alpha, \alpha \rangle} \). Then \( \sigma_\alpha(x) = \langle \alpha^\vee, x \rangle \alpha \)

Definition 12 A root system is a collection \( \Phi \) of nonzero vectors (called roots) in a real vector space \( V \) such that:

1. \( \Phi \) is finite,
2. \( 0 \notin \Phi \) and \( \Phi \) spans \( V \),
3. For each root \( \beta \), the reflection \( \sigma_\beta \) permutes \( \Phi \),
4. Given a line \( L \) through the origin, either \( L \cap \Phi \) is empty or \( L \cap \Phi = \{\pm \beta\} \) for some \( \beta \) (reduced system condition),
5. \( \langle \alpha^\vee, \beta \rangle \in \mathbb{Z} \), for each \( \alpha, \beta \in \Phi \) (crystallographic condition).

Definition 13 Two root systems \( \Phi \subset V \) and \( \Phi' \subset V' \) are isomorphic if there is an isometry \( f : V \to V \) with \( f(\Phi) = \Phi' \).

Exercise 13 Describe all not necessarily reduced finite one-dimensional crystallographic root systems up to an isomorphism.

Exercise 14 Let \( \theta \) be an angle between vectors \( \alpha, \beta \). Show that \( \langle \alpha^\vee, \beta \rangle\langle \beta^\vee, \alpha \rangle = 4\cos^2 \theta \) and find possible values of \( \theta \), \( \langle \alpha^\vee, \beta \rangle \), \( \langle \beta^\vee, \alpha \rangle \) and \( 4\cos^2 \theta \) for vectors in a finite crystallographic root system.

Exercise 15 Let \( \alpha, \beta \) be two non proportional vectors in a finite crystallographic root system \( \Phi \). Show that if \( \langle \alpha, \beta \rangle < 0 \) then \( \alpha + \beta \in \Phi \). If \( \langle \alpha, \beta \rangle > 0 \) then \( \alpha - \beta \in \Phi \).
Definition 14 Let \( \alpha, \beta \) be a pair of linearly independent roots. A subset \( \{ \gamma \in \Phi | \gamma = \beta + k\alpha (k \in \mathbb{Z}) \} \) of a root system \( \Phi \) is called an \( \alpha \)-series of roots, containing \( \beta \). In particular if \( \beta - \alpha \notin \Phi \) then \( \beta + \alpha \in \Phi \) iff \( \langle \beta, \alpha \rangle < 0 \).

Exercise 16 An \( \alpha \)-series of roots, containing \( \beta \) has a form \( \{ \beta + k\alpha | -p \leq k \leq q \} \), where \( p, q \geq 0 \) and \( p - q = \langle \alpha^\vee, \beta \rangle \).

Definition 15 Exercise 17 We define a collection \( \Phi^\vee = \{ \alpha^\vee | \alpha \in \Phi \} \subset V \). Prove that \( \Phi^\vee \) is a root system.

(Direct sums). Let \( \Phi \) and \( \Phi' \) be root systems in \( V \) and \( V' \), respectively. Then \( \Phi \cup \Phi' \) is a root system in the vector space \( V \oplus V' \). A root system is reducible if it can be written as such an (orthogonal) direct sum, and irreducible otherwise.

Definition 16 Let \( \Phi \) be a root system. Then the Weyl group of \( \Phi \) is the group generated by \( \sigma_\alpha \) for all \( \alpha \in \Phi \).

Exercise 18 Is the Weyl group well-defined (i.e., do isomorphic root systems give isomorphic Weyl groups?).

Exercise 19 Is the Weyl group of a finite root system finite?

Exercise 20 What are the Weyl groups of the four crystallographic root systems in \( \mathbb{R}^2 \)?

Exercise 21 Find a root system having the symmetric group on four letters, \( S_4 \), as its Weyl group.

Definition 17 Let \( \Phi \subset V \) be a root system, and choose \( v \in V \). Define \( \Phi^+(v) = \{ \alpha \in \Phi | \langle \alpha, v \rangle > 0 \} \). We say that \( v \) is regular if \( \Phi = \pm \Phi^+(v) \), and singular otherwise. If \( v \) is regular, we call \( \Phi^+(v) \) a positive system for \( \Phi \).

Exercise 22 Why does a regular \( v \) exist?

Let \( v \) be regular we set \( \Phi^+ = \Phi^+(v) \). In general \( \Phi^+ \) depends on the choice of \( v \).

Definition 18 The set \( \Pi(\Phi^+) \subset \Phi^+ \) is formed by elements \( \alpha \) that can not be presented as a sum \( \alpha = \beta \gamma, \beta \gamma \in \Phi^+ \).
Exercise 23  Show that any $\alpha \in \Phi^+$ can be written in the form $\alpha = \sum_{\beta \in \Pi(\Phi^+)} c_{\beta} \beta$, where $c_{\beta}$ are nonnegative integers.

Exercise 24  If $\alpha, \beta \in \Pi(\Phi^+)$ and $\alpha \neq \beta$, then $\alpha - \beta \neq \Phi$ and $\langle \alpha, \beta \rangle \leq 0$.

Exercise 25  Let $\alpha_1, \ldots, \alpha_k$ be a set of vectors in $V$ such that $\langle \alpha_i, \alpha_j \rangle \leq 0, i \neq j$. Suppose we have a nontrivial linear combination with positive $c_i, c'_j$:

$$\sum_{r=1}^{k} c_r \alpha_{i_r} - \sum_{r'=1}^{l} c'_{r'} \alpha_{j_{r'}} = 0$$

with all $i_1, \ldots, i_k, j_1, \ldots, j_l$ distinct. Then

1. $\sum_{r=1}^{k} c_r \alpha_{i_r} = \sum_{r'=1}^{l} c'_{r'} \alpha_{j_{r}} = 0$.
2. $\langle \alpha_{i_r}, \alpha_{j_{r'}} \rangle = 0, r = 1, \ldots, k, r' = 1, \ldots, l$

Exercise 26  Let $\alpha_1, \ldots, \alpha_k \in V$ be a set of linearly independent vectors. Show that there is $\beta \in V$ such that $\langle \alpha_i, \beta \rangle > 0, i = 1, \ldots, k$

Definition 19  The $n$-th Catalan number $C_n$ is the number of full binary planar trees with $n + 1$ leaves.

Exercise 27  Prove the formula

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

Exercise 28  Prove the Ptolemy’s theorem: let $\Delta_{ABCD}$ be a quadrilateral whose vertices lie on a common circle. Then

$$|AC||BD| = |AB||CD| + |BC||AD|$$