Selected practice exam solutions (part 5, item 2) (MAT 360)

Harder 82,91,92,94(smaller should be replaced by greater)95,103,109,140,160,(178,179,180,181 this is really one problem),188,193,194,195

82. On one side of an angle A, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ meet on the bisector of the angle A.

The reflection in the bisector exchanges the rays forming the angle. Since the reflection sends A to A, the congruence $AB' = AB$ implies that B is sent to B' and the congruence $AC' = AC$ implies that B is sent to B'. The line $B'C$ is sent to the line BC', and vice-versa. The intersection point of $B'C/BC$, call it O, is sent to the intersection point of $BC'/B'C$, which is the same point. Then O lies on the line of symmetry of the reflection, the bisector of the angle A.

91. A median drawn to a side of a triangle is smaller than the semisum of the other two sides.

The triangle has sides a, b, c, and the median bisects c. Following the hint, double the median by prolonging it past the side it bisects. The segments drawn from the new endpoint to the original vertices are also a and b by the Side Angle Side congruence test. Then by the triangle inequality,

$$2m < a + b$$

$$m < (a + b)/2$$

92. The sum of the medians of a triangle is smaller than its perimeter but greater than its semi-perimeter.

According to exercise 90 (homework 2), each median of a triangle is smaller than its semiperimeter. By addition, we know that the sum of the medians is smaller than 3/2 of the perimeter, but according to this exercise this sum is actually even smaller than the perimeter itself. This follows by adding up the 3 inequalities in exercise 91. Let x, y, z be the medians cutting sides a, b, c. Then

$$
\begin{aligned}
2x &< b+c \\
2y &< a+c \\
2z &< a+b \\
x+y+z &< a+b+c
\end{aligned}
$$

This is an upper bound. For the lower bound:
Again let the medians be x, y, z corresponding to sides a, b, c. For each median, select a side of the original triangle which is smaller (by choosing the side across from the acute angle formed by the median at the point of bisection). The list of 3 sides obtained this way cannot be the full list $\{a, b, c\}$, since then the sum of the medians would be larger than the perimeter. Then at least one side is missing from this list; suppose that c is missing.

$$x+y+z < a+a+b = a+(a+b) < a+c = \frac{a+c}{2} + \frac{a+c}{2} < \frac{a+c}{2} + \frac{b}{2} = \frac{a+b+c}{2}$$

94. The sum of segments connecting a point inside a triangle with its vertices is less than the semiperimeter of the triangle.

Let x, y, z be the segments emanating toward sides a, b, c. According to the triangle inequality,

$$
\begin{aligned}
x+y &< c \\
x+z &< b \\
y+z &< a
\end{aligned}
$$

By addition, $2(x+y+z) < a+b+c$, that is $x+y+z < (a+b+c)/2$.

95. Given an acute angle XOY and an interior point A. Find a point B on the side OX and a point C on the side OY such that the perimeter of the triangle ABC is minimal.

The construction is exercise 219. For an illustration, see the solution page for exercises 213-224.
First, let's see that if ABC is minimal, then $OB = OC$. If not, assume $OB < OC$, and let B' and C' be the reflections in the bisector of angle XOY. Note that B and C are both on the same side as O, the same side of the perpendiculars from A to the angle sides; otherwise all 3 sides of triangle ABC can be decreased by selecting new B and C closer to O.
Then angle $OBA = OB'A$ is obtuse. According to the inequalities related to slants, sides $BC = B'C'$ and $BA = B'A'$ will decrease if new B and B' are selected between the old B, B' and C, C'. The third side of the triangles ABC and $AB'C$ will remain the same. This is a contradiction, so $OB = OC$ for any minimal solution.

(Rest of proof: Prove that $OB = OC$ and ABC minimal perimeter implies that angles OAB and OAC are congruent to the original angle XOY?)

103. A line and a circle can have at most 2 common points.

Suppose that a line and a circle both contain 3 distinct points X, Y, Z, and assume that Y is between X and Z. Let C be the center of the circle. The triangles XCY, XCZ, YCZ are all isosceles with vertex C, so the angles CYX and CYZ are congruent. That means CYX and CYZ are both half a straight angle, and therefore all the angles formed at the points X, Y, Z between the given line and the ray toward C are right. No triangle has 2 right angles.
This proves that a line and a circle can commonly contain no more than 2 points.

109. Find the geometric locus of vertices A of triangles ABC with the given base BC and $\angle B > \angle C$.

This exercise appears in homework 3.
Pick a point A not on the line BC. If A lies on the perpendicular bisector of BC, then triangle ABC is isosceles, so angle B and angle C are equal.
Otherwise, A lies on one or the other side of the perpendicular bisector. If it lies on the same side as B, then it lies on the opposite side from C, so line AC meets the perpendicular at a point X between A and C. BX and CX are congruent, since points along the perpendicular bisector of a segment are equidistant from the endpoints. Angle XBC is contained in angle ABC, so it is lesser. But angle XBC is congruent to ACB, by Side Angle Side triangle congruence $XBC = XCB$. So angle B is greater than angle C.
Similarly, were A picked on the other side of the perpendicular, the roles would be reversed, and angle C would be greater than B.
Therefore the geometric locus in question is all points, not lying on the line BC or its perpendicular bisector, lying on the same side of this bisector as C.

140. Divide the plane by infinite straight lines into five parts, using as few lines as possible.
The plane can be divided in $n + 1$ parts by n parallel lines, so we can divide it in 5 parts with 4 parallel lines. Can we do it with fewer lines?
No. Consider a configuration with 3 or fewer lines. If all are parallel, there are only 3 regions formed. Then some pair intersect. If there are only these 2 lines, then 4 regions are formed (which is not 5). Then there must be a third line. If this third line is parallel to one or the other of the first two, then 6 regions are formed. Then, instead, the third line must be skew to both other lines. If the two resulting intersection points are distinct from each other and from the remaining intersection, then these 3 intersection points cannot be collinear; if they were, all lines would be the same. Instead, these 3 points are non-collinear, and hence they form a triangle, and in this case $3 + 3 + 1 = 7$ regions are formed.

160. Prove that in a convex polygon, one of the angles between the bisectors of two consecutive angles is congruent to the semisum of these two angles.

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(178,179,180,181 this is really one problem)
178. Midpoints of the sides of a quadrilateral are the vertices of a parallelogram. Determine under what conditions this parallelogram will be (a) a rectangle (b) a rhombus (c) a square.
179. In a right triangle, the median to a hypotenuse is congruent to a half of it.
180. Conversely, if a median is congruent to a half of the side it bisects, then the triangle is right.
181. In a right triangle, the median and the altitude drawn to the hypotenuse make an angle congruent to the difference of the acute angles.

There are two basic facts. First, the altitude of a right trangle drawn to the hypotenuse divides the right angle into the same angles as the acute angles of the original triangle. This is because two new right triangles are formed, and right triangles have complementary acute angles.
Second, the median of a right triangle divides the triangle into isosceles triangles whose bases are the original legs. For, consider one of these new triangles, T. Its altitude dropped from the original hypotenuse midpoint is parallel to a leg of the original triangle. Parallel lines cutting out congruent segments on one line cut out congruent segments on all lines, so the foot of this altitude divides a leg of the original triangle in half. By the Side Angle Side congruence theorem, this altitude divides T into congruent triangles, making T isosceles. By inspecting which two sides are congruent as a result, this already proves 179. For 180, the converse: If the median is congruent to half of the side it bisects, it divides the triangle into two isosceles triangles T and T'. The vertex angles of are supplementary, which implies that the base angles are complementary, which implies that the triangle is right.
Combining these two facts, the angle in question–between the median and altitude drawn to the altitude of a right triangle–is contained in an angle congruent to one of the acute angles A of the right triangle, such that the remainined angle is congruent to the other acute angle B; it is congruent to the difference of these angles. This establishes 181.
Finally for 178: Consider the parallelogram formed from the midpoints of a quadrilateral. Recall the classification of quadrilaterals (homework 3) by number of axes of symmetry. This classification can be refined by number and degree of rotational symmetries, or other properties (e.g. convexity). In this case the crudest classification, by number of axes and angle type, gives the whole story:
4 axes of symmetry (square): In this case the associated parallelogram is evidently a square; indeed, the associated parallelogram will have at least as many axes of symmetry as the original quadrilateral, and the only quadrilateral having 4 axes of symmetry is the square.
2 axes of symmetry (rectangle or rhombus): In the case of a rectangle or a rhombus, the diagonals of the associated parallelogram are congruent to the original sides. Use the previous results for the triangles formed by casting segments from the diagonals' intersection to the original vertices: in the rectangle case, since half of one side is not congruent to half of the other side, the angles of the associated parallelogram are not right. Since this parallelogram nevertheless has 2 axes of symmetry, it is a rhombus. Similarly, the associated parallelogram of a rhombus is a rectangle.
1 axis of symmetry ("kite" or "special trapezoid"): The associated parallelogram has an axis of symmetry, so it is therefore a rectangle or a rhombus. Indeed, in the "kite" case, the axis

of symmetry passes through 2 original vertices, so the associated parallelogram has an axis of symmetry not passing through the vertices; it is a rectangle. In the "special trapezoid" case, the line of symmetry passes through two midpoints, and hence through two vertices of the associated parallelogram, which is therefore a rhombus.

188. 187 says, in an isosceles triangle, the sum of the distances from each point of the base to the lateral sides is constant, namely it is congruent to the altitude dropped to a lateral side. How does this theorem change if the points on the extension of the base are taken instead?

If the point of the base lies on the part of the base extension which is opposite the side x of the isosceles triangle, then what is constant now is the the distance to the nearer lateral side extension *subtracted from* the distance to the side x. This is proved by reproducing the isosceles triangle rotated a complete straight angle around a vertex on the base; the perpendiculars dropped from the base point to the far side and to it's corresponding side on the rotated triangle will coincide. Then the two distances from the base point along this perpendicular will sum to a constant, namely twice an altitude of the original triangle. We combine this fact with the exercise 187, applied to both triangles, and we have the result.

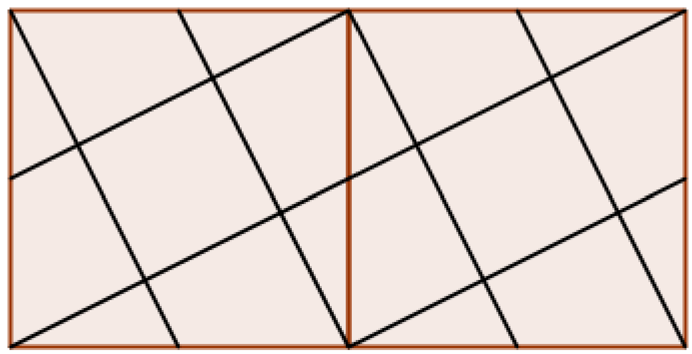
193. From the intersection point of the diagonals of a rhombus, perpendiculars are dropped to the sides of the rhombus. Prove that the feet of these perpendiculars are vertices of a rectangle.

The diagonals of a rhombus are lines of symmetry of the rhombus. Therefore they are lines of symmetry of the altitudes dropped from the diagonal intersection to the sides. Similarly, they are lines of symmetry of the quadrilateral formed from the resulting 4 feet. Since the original diagonals are perpendicular, and they do not contain the 4 feet, the quadrilateral formed must be a rectangle (by the classification of quadrilaterals, e.g. as in homework 3).

194. Bisectors of the angles of a rectangle cut out a square.

The bisectors of the two angles at a smaller side of a rectangle meet at a right angle, since this point is the vertex of an isosceles triangle with half-right base angles. Doing this for both smaller sides, we obtain 2 congruent isosceles right triangles. Translating them to have a common hypotenuse, the resulting quadrilateral has equal sides and all angles are right; it is a square.
The words "cut out" probably refer to the fact that the triangles constructed above lie inside the rectangle, whereas they would not if we used the larger sides instead. This fact require proof, but it also requires more precise definitions of "lie inside".

195. Let A', B', C', and D' be the midpoints of the sides CD, DA, AB, and BC of a square. Prove that the segments AA', CC', DD', and BB' cut out a square, whose sides are congruent to 2/5th of any of the segments.



We are supposed to think of the smaller, triangular pieces being excised; we are to show that the remaining piece in the center is a square of side 2/5 of the cutting segments.
The shape is evidently a quadrilateral, which is rotationally symmetric of order 4. Therefore it is a square. Indeed, the picture suggests a clear argument. Looking at the large diagonal, we should show that the lower left segment, the upper right segment, and the middle segment joining two smaller segments are all congruent to the square side. Then 2 of the cut segments are congruent to 5 of the square segments.
The same argument will work for the upper right and lower left parts: parallel lines cutting one line in equal parts–namely the original square–cut all other lines in the same ratio. As for the middle two small segments, we find that their sum is the side of the the interior square of a square congruent to the original one: the square formed from sufficiently many translates of the lower-left interior square vertex.