REVIEW FOR MAT 341 MIDTERM II
NOVEMBER 2018

Exam covers §§2.4.3, 2.4.5.-2.4.7, 2.5.1-2.5.5, 5.1.1-5.1.3, 5.2.1-5.2.2, 5.2.4-5.2.6.

1. The wave equation on the interval $0 < s < L$. Solution of the vibrating string problem by separation of variables (§§2.4.3 and 2.4.6).

2. Representation of the solution of the vibrating string in terms of odd $2L$-periodic extensions $\bar{f}_1$ and $\bar{f}_2$ of the initial data f_1 and f_2 (§§2.4.5 and 2.4.6):

$$y(s,t) = \frac{1}{2}(\bar{f}_1(s+ct) + \bar{f}_1(s-ct)) + \frac{1}{2c}\int_{s-ct}^{s+ct} \bar{f}_2(z)dz,$$

or, equivalently

$$y(s,t) = \frac{1}{2}(\bar{f}_1(s+ct) + \bar{f}_1(s-ct)) + \frac{1}{2}(\bar{G}(s+ct) - \bar{G}(s-ct)),$$

where $\bar{G}(s)$ is an even $2L$-periodic extension of the function $G(s)$,

$$G(s) = \frac{1}{c}\int_0^s f_2(z)dz, \quad 0 < s < L.$$

3. Reading the properties of the motion of the plucked string from the graph of $f_1(s)$, sketching the graph of $y(s,t)$ on $0 < x < L$ for different values of t (§2.4.5). Same for the struck string (§2.4.6).

4. D'Alembert solution (§2.4.7).

5. Double Fourier sine series. Solution of the initial-value problem for the two-dimensional heat equation

$$u_t = K(u_{xx} + u_{yy})$$

in the column $0 < x < L_1,\ 0 < y < L_2$ with the homogeneous Dirichlet boundary conditions and the initial condition

$$u(x,y,0) = f(x,y).$$

(See §2.5.1). Other homogeneous boundary conditions (see problems 1-2 in the HW 9).

6. The two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0$$

in the rectangle $0 < x < L_1,\ 0 < y < L_2$. Solution of the boundary value problem with boundary conditions

$$\begin{aligned} u(x,0) &= f_1(x), \quad 0 < x < L_1, \\ u(x,L_2) &= f_2(x), \quad 0 < x < L_1, \\ u(0,y) &= g_1(y), \quad 0 < y < L_2, \\ u(L_1,y) &= g_2(y), \quad 0 < y < L_2 \end{aligned}$$

by using representation

$$u(x,y) = u_1(x,y) + u_2(x,y).$$

Here u_1 is the solution of the Laplace equation for the case $g_1(y) = g_2(y) = 0$, $0 < y < L_2$, obtained by separation of variables and solving the Sturm-Liouville eigenvalue problem on $0 < x < L_1$ with DBC, and matching the remaining BC at horizontal sides $y = 0$ and $y = L_2$, and u_2 is the solution of the Laplace equation for the case $f_1(x) = f_2(x) = 0$, $0 < x < L_1$, obtained by separation of variables and solving the Sturm-Liouville eigenvalue problem on $0 < y < L_2$ with DBC, and matching the remaining BC at vertical sides $x = 0$ and $x = L_1$. (See examples in §2.5.2). According to the general scheme presented in class,

$$\begin{aligned} u_1(x,y) &= \sum_{n=1}^{\infty} \sin \mu_n x \left(A_n \frac{\sinh \mu_n (L_2 - y)}{\sinh \mu_n L_2} + B_n \frac{\sinh \mu_n y}{\sinh \mu_n L_2} \right), \quad \mu_n = \frac{\pi n}{L_1}, \\ u_2(x,y) &= \sum_{n=1}^{\infty} \sin \nu_n y \left(C_n \frac{\sinh \mu_n (L_1 - x)}{\sinh \nu_n L_1} + D_n \frac{\sinh \nu_n x}{\sinh \nu_n L_1} \right), \quad \nu_n = \frac{\pi n}{L_2}, \end{aligned}$$

where A_n and B_n are Fourier sine coefficients of the functions $f_1(x)$ and $f_2(x)$ on $0 < x < L_1$, and C_n and D_n are Fourier sine coefficients of the functions $g_1(y)$ and $g_2(y)$ on $0 < y < L_2$.

7. Solving the initial-value problem for the two-dimensional heat equation with non-homogeneous BC and initial condition $f(x,y)$ (see §2.5.3). Namely,

(i) solve for the steady-state solution $U(x,y)$ — solution of the Laplace equation with BC as in **6**;

(ii) solve the initial-value problem for the transient solution

$$v(x,y,t) = u(x,y,t) - U(x,y)$$

of the two-dimensional heat equation with homogeneous BC and initial condition $v(x,y,0) = f(x,y) - U(x,y)$ — compute Fourier double sine series of the function $f(x,y) - U(x,y)$ (see p. 158 in the textbook) and use the method in **5**;

(iii) write the solution as

$$u(x,y,t) = v(x,y,t) + U(x,y).$$

8. Solution of the three-dimensional Laplace equation in the cube and double Fourier sine series (§2.5.2).

9. Definition and properties of the Fourier transform (§§5.1.1–5.1.3):
   (i) Fourier inversion formula (Theorem 5.1);
   (ii) basic properties of the Fourier transform (Theorem 5.2);
   (iii) textbook and homework examples;
   (iv) Fourier cosine and sine transforms.

10. The heat equation for an infinite rod (§§5.2.1–5.2.2 and 5.2.4). Solution using Fourier transform — formulas (5.2.3)–(5.2.4), and explicit representation using Cauchy-Weierstrass kernel — formula (5.2.10). Examples in the homework and in the textbook (§5.2.5).

11. Solution of the heat equation on the half-line using the method of images (§5.2.6) and Fourier cosine and sine transform (homework problems).