MAT313 Fall 2017
Practice Midterm II

The actual midterm will consist of 5-6 problems.

It will be based on subsections 127-234 from sections 6-11. This is a preliminary version. I might add few more problems to make sure that all important concepts and methods are covered.

Problem 1.

Let $\mathbf{P}$ be a set of lines through 0 in $\mathbb{R}^2$. The group $\mathrm{SL}(2, \mathbb{R})$ acts on X by linear transformations. Let H be the stabilizer of a line defined by the equation $y = 0$.

(1) Describe the set of matrices H.
(2) Describe the orbits of H in X. How many orbits are there?
(3) Identify X with the set of cosets of $\mathrm{SL}(2, \mathbb{R})$.

Problem 2.

(1) Describe all elements of order eight in $\mathbb{Q}/\mathbb{Z}$.
(2) Find all elements of infinite order in $\mathbb{Q}/\mathbb{Z}$.
(3) Identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$

Problem 3. Let G be the group of quaternions, i.e., $G = \{1, -1, i, j, k, (-1)i, (-1)j, (-1)k\}$. The elements $1, -1$ are central and satisfy $-1^2 = 1$. In addition $i^2 = j^2 = k^2 = -1$, $(-1)ji = ij = k, (-1)ki = ik = (-1)j, (-1)jk = kj = (-1)i$.

Find orders of all elements in $G/Z(G)$, Is $G/Z(G)$ isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. Why?

Problem 4. Give the definition of a factor group.

Problem 5. Fix a group G. Prove that $|x| = |gxg^{-1}| \forall x, g \in G$. Deduce that $|xy| = |yx| \forall x, y \in G$.

Problem 6. Compute the order of $GL_2(\mathbb{Z}_p)$ where p is a prime number.

Problem 7.

(1) Prove that dihedral group D_{12} (the group of symmetries of the regular 12-gon) is not isomorphic to symmetric group S_4.
(2) Prove that dihedral group $\mathbb{Z} \neq \mathcal{Q}$.

A group G action on the set X is faithful if the corresponding homomorphism $\rho : G \to S_X$ has a trivial kernel.

Problem 8. Prove that the group of rigid symmetries of a cube doesn't act faithfully on the set of opposite faces of the cube. Find the kernel.

Problem 9. Give an example of a noncommutative group G and a normal subgroup $N \subset G$ such that G/N is abelian of order ≥ 3.

Problem 10. Let G be a finite group, let H be a subgroup and $N \trianglelefteq G$. Prove that if $\gcd(|H|, |G : N|) = 1$, then $H \subset N$.

Problem 11. Prove that if H has a finite index n in G then there is $N \trianglelefteq G$ such that $|G : N| \leq n!$

Problem 12. Let $A = \mathbb{Z}_{60} \times \mathbb{Z}_{45} \times \mathbb{Z}_{12} \times \mathbb{Z}_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.