MAT313 Fall 2013

Practice Midterm II

The actual midterm will contain five problems

Problem 1. Do noninvertible elements form an ideal in

(1) $\mathbb{Z}_n$

(2) $\mathbb{Z}$

Solution. (1) Yes. if $a \in \mathbb{Z}_n$ isn't invertible $\Rightarrow (a, n) > 1 \Rightarrow (ka, n) > 1$

(2) No. Among noninvertible elements we have relatively prime elements a, b. $\exists s, t\ as + bt = 1$. The unit is invertible $\Rightarrow$ noninvertible elements is not an ideal.

□

Problem 2. Prove that the number of units in $\mathbb{Z}[\sqrt{3}]$ is infinite

Solution. We have a unit $2+\sqrt{3}$. It is a unit because $(2+\sqrt{3})(2-\sqrt{3}) = 4-3 = 1$. A product of units is is a unit $\Rightarrow (2+\sqrt{3})^n$ is a unit. Equality $(2+\sqrt{3})^n = 1$ is impossible because $2+\sqrt{3} > 1$. □

Problem 3.

Prove that the ring $R = \{a + b\sqrt[3]{7} + c\sqrt[3]{49} | a, b, c \in \mathbb{Q}\}$ is a field.

Solution. There is an onto map $\phi : \mathbb{Q}[x] \to R$, $\phi(p(x)) = p(\sqrt[3]{7}), p(x) \in \mathbb{Q}[x]$. The ring R is a field iff $\mathrm{Ker}\phi$ is maximal. Since all ideals in $\mathbb{Q}[x]$ are principal $\Rightarrow$ $\mathrm{Ker}\phi = \langle g \rangle$, for $g \in \mathbb{Q}[x]$. It is clear that $x^3 - 7 \in \mathrm{Ker}\phi \Rightarrow g | x^3 - 7$. $7 \in \mathbb{Q}$ not a cube $\Rightarrow g = a(x^3 - 7), a \in \mathbb{Q}$. Ideal $\langle x^3 - 7 \rangle$ is maximal because as we know if $\langle x^3 - 7 \rangle \subset I = \langle h \rangle \Rightarrow h | x^3 - 7 \Rightarrow h = a(x^3 - 7)\ a \in \mathbb{Q} \Rightarrow R$ is a field. □

Problem 4.

Find all solutions of $x^2 - x + 2 = 0$ in $\mathbb{Z}_7[i]$.

Solution. With a help of the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for solutions of $ax^2 + bx + c = 0$ we get

$$x = \frac{1 \pm 3i}{2} = 4 \pm 5i$$

□

Problem 5.

Show that $\mathbb{R}[x]/(x^2-2)$ is not a field, but $\mathbb{Q}[x]/(x^2-2)$ is.

Solution. (1) Ideal $\langle x^2-2\rangle \subset \mathbb{R}[x]$ is not maximal because $\langle x^2-2\rangle \subset \langle x-\sqrt{2}\rangle \subset \mathbb{R}[x] \Rightarrow \mathbb{R}[x]/(x^2-2)$ is not a field

(2) x^2-2 is irreducible in $\mathbb{Q}[x]$ because $2 \in \mathbb{Q}$ is not a square.$\Rightarrow \langle x^2-2\rangle \subset \mathbb{Q}[x]$ is maximal $\Rightarrow \mathbb{Q}[x]/(x^2-2)$ is a field.

□

Problem 6.

Find all the ring homomorphisms $\psi : \mathbb{R} \to \mathrm{Mat}_2(\mathbb{R})$ and $\phi : \mathrm{Mat}_2(\mathbb{R}) \to \mathbb{R}$. ψ and ϕ are homomorphisms of rings with units, i.e. $\psi(1) = 1$ and $\phi(1) = 1$, which are $\mathbb{R}$-linear maps.

Solution. (1) ψ is completely determined by the condition $\psi(1) = 1 = \mathrm{Id}$, where Id is the identity matrix. The homomorphism condition $\psi(a)\psi(b) = a\mathrm{Id}ba\mathrm{Id} = ab\mathrm{Id}$ trivially holds.

(2) Let $\phi : \mathrm{Mat}_2(\mathbb{R}) \to \mathbb{R}$ be some homomorphism of algebras. If $\phi \neq 0$, then it is onto. The ring $\mathrm{Mat}_2(\mathbb{R})$as a vector space over $\mathbb{R}$ has dimension 4. $\Rightarrow$ $\dim \mathrm{Ker}\phi = 3$. I claim that $\mathrm{Mat}_2(\mathbb{R})$ contains no nontrivial ideals. Indeed let e_{ij} be a matrix with a unit on (i, j) slot. in if $a\mathrm{Ker}\phi \Rightarrow e_{ii}ae_{jj} \in \mathrm{Ker}\phi$. If we choose i, j in such a way that $a_{ij} \neq 0 \Rightarrow e_{ii}ae_{jj}/a_{ij} = e_{ij}$. We can use elementary matrix transformations to transform e_{ij} into any $e_{i'j'}$. We know from linear algebra that elementary matrix transformations are obtained by left and right multiplication on elementary matrices. $\Rightarrow e_{i'j'} \in \mathrm{Ker}\phi$ Since $\{e_{i'j'}\}$ is a basis in $\mathrm{Mat}_2(\mathbb{R}) \Rightarrow \mathrm{Ker}\phi = \mathrm{Mat}_2(\mathbb{R})$.

□

Problem 7. Do homomorphisms of a ring form a group with respect to addition. Explain.

Solution. No. In general if ϕ_1, ϕ_2 are homomorphisms then $\phi = \phi_1 + \phi_2$ doesn't satisfy $\phi(xy) = \phi(x)\phi(y)$:

$$\phi(xy) = \phi_1(xy)+\phi_2(xy) \neq (\phi_1+\phi_2)(x)(\phi_1+\phi_2)(y) \neq \phi_1(x)\phi_1(y)\phi_2(x)\phi_2(y)+\phi_2(x)\phi_1(y)+\phi_1(x)\phi_2(y)$$

$\square$