Solution 0.1 *(problem 1) You should take this problem as a list of definitions you should know. Consult the book if you don't know it.*

Solution 0.2 *(problem 3) In matrix form, the problem is finding x in:*

$$Ax = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & 1 & -1 & 3 \\ 5 & 0 & 1 & 5 \end{pmatrix}$$

Row reduce A to find the solutions.

Solution 0.3 *(problem 5) $P^2 = P$ tells us the eigen values of P are all 1 or the vector is in the kernel. Take the basis in these eigen values.*

Solution 0.4 *(problem 7)*

1. *TRUE - check axioms.. closed under addition mult. contains zero. etc.*
2. *TRUE - composition of linear functions of scalar multiplication, addition and differentiation.*
3. *TRUE - all that is needed though is the determinant $\neq 0$*
4. *TRUE - a non-zero off diagonal entry a_{ij} would make the image of e_j have a non-zero e_i component. Contradicting e_j being an eigen-vector.*
5. *TRUE - assuming the book allows eigen values to be zero.*
6. *TRUE - conjugate by the matrix S which sends the eigen vectors of one to the eigen vectors of the other.*

Solution 0.5 *(problem 9) Diagonalize S and split the vector space into eigen spaces W_λ. Commutativity implies that the sub space W_λ is spanned by the eigen-vectors of S with eigen-value λ are preserved by T:*

$$T(W_\lambda) \subset W_\lambda$$

This follow from: $Sv = \lambda v \Rightarrow S(Tv) = \lambda Tv$

Since T diagonalizes on the whole space, it also diagonalizes on W_λ for some possible distinct different basis. Conjugate S into this new basis. Since $S = \lambda Id$ on W_λ, S is still diagonal.

Doing this on W_λ $\forall\lambda$ gives us our result.

Solution 0.6 *(problem 11) Let's look at the transpose, A^t as invertibility of the transpose is the same as invertibility of A. Invertibility is equivalent to injectivity in this case.*

Let $p(x)$ be a degree $n-1$ polynomial with coefficients: $p_0, ..., p_{n-1}$. Then:

$$A \cdot \begin{pmatrix} p_0 \\ \vdots \\ p_{n-1} \end{pmatrix} = \begin{pmatrix} p(\lambda_1) \\ \vdots \\ p(\lambda_n) \end{pmatrix}$$

by the hint, at least one of these is non-zero.

Since any vector has an interpretation as the coefficients of a polynomial in $n-1$ variables, none of them are in the kernel.

Solution 0.7 *(problem 13) The idea here is to take expand along the rows columns with as many zeros as possible. Brute forcing it is doable, but will waste time and leave openings for mistakes,*

1. *Expand along the last column:*

$$2 \times \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 5 & 0 & 3 \\ 0 & 3 & 0 & 2 \\ 5 & 2 & -1 & 1 \end{pmatrix} - (-1) \times \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 5 & 0 & 3 \\ 0 & 3 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2. *Expand along the second to last column in each of these*

$$2 \times (-1) - 1 \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 2 \end{pmatrix} - (-1) \times (-1)1 \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

3. *etc....*

Solution 0.8 *(problem 15) The characteristic polynomial factors to:*

$$(t-7)(t-1)$$

This means their is a basis (eigen-vectors) in which the matrix can be written as:

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

Now A' is similar to A:

$$A = SA'S^{-1}$$

so we will be done if we can compute S.

Well,

$$\begin{pmatrix} 41 & -40 \\ -8 & 9 \end{pmatrix} = A^2 = \left(SA'S^{-1}\right)^2$$
$$= S\left(A'\right)^2 S^{-1} = S \begin{pmatrix} 1 & 0 \\ 0 & 49 \end{pmatrix} S^{-1}$$

Now S takes the eigen vectors of A' $\{e_1, e_2\}$ to the eigen vectors of A. So the column vectors of S are given by the eigen vectors of A^2.

By solving for:

$$A^2 \begin{pmatrix} x \\ y \end{pmatrix} = 1 \times \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$A^2 \begin{pmatrix} x \\ y \end{pmatrix} = 49 \times \begin{pmatrix} x \\ y \end{pmatrix}$$

We get the eigen vectors