MAT310 Fall 2012

Practice Final

The actual Final exam will consist of twelve problems that cover chapters 1.2-5.4 (inclusive) with omission of 2.6,4.5,5.3

Problem 1 Let V, W be vector spaces. Define the following terms:

1. What is a subspace of V ?

2. Let $F : V \to W$ be a function. What does it mean to say that F is linear?

3. Let $T = \{v_1, v_2, \dots\}$ be a subset of V . What is a linear combination of elements of T? What is the span of T? What does it mean to say that T is linearly independent? What does it mean to say that T spans V ? What does it mean to say that T is a basis of V ?

4. What is the dimension of V ?

5. Let $F : V \to W$ be linear. Define $N(F) = ker(F)$. Define $im(F)$. What is the *rank* of F? What is the nullity of F?

6. Let $F : V \to V$ be linear. What is an eigenvalue of F? What is an eigenvector of F?

7. What does it means to say that two $n \times n$ matrices are similar?

8. What does it mean to say that two vector spaces are isomorphic?

9. Let A be an $n \times n$ matrix. What is an eigenbasis for the matrix A?

10. Let B be a basis of a vector space V . What does one mean by the coordinates of a vector $v \in V$ with respect to B?

Problem 2

1. Let $F : V \to W$ be linear. Show that $ker(F)$ is a subspace of V.
   Show that $im(F)$ is subspace of W

2. State the rank+nullity theorem.

Problem 3 Consider the system of equations

$$\begin{cases} x - 2y + 3z - w = 2 \\ 2x + y - z + 3w = 1 \\ 5x + z + 5w = 4 \end{cases}$$

1. Find all, if any, solutions to this system.

2. Write the system as a matrix equation

Problem 4 Determine linearly independent sets

1. Set of functions 1, e^x and e^{2x} thought of as elements of real linear space of continuous functions $C[0,1]$

2. Set of functions 1, $\sin^2(x)$ and $3 - \cos^2(x)$ thought of as elements of real linear space of continuous functions $C[0, 2\pi]$

Problem 5 Let $P : V \to V$ be a projection on a finite dimensional vector space, i.e., P is a linear map with the property that $P^2 = P$. Show that there exists a basis B for V such that $M(P) = \left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$,where I_r is the $r \times r$ identity matrix. (Here $M(P)$ is the matrix representing the map P relative to the basis B.)

Problem 6 Find bases in *ImA* and *kerA* where the linear transformation $A : \mathbb{R}^5 \to \mathbb{R}^3$ has a matrix

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 0 \\ 2 & 1 & 1 & -1 & 2 \\ 5 & 0 & 3 & 0 & 4 \end{bmatrix}$$

Extend the bases to bases of $\mathbb{R}^3$, $\mathbb{R}^5$ respectively.

Problem 7 True or False. (Explain!)

1. The set of all vectors of the form $(a, b, 0, b)^t$ where a, b are real numbers forms a subspace in $\mathbb{R}^4$.
2. Let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have infinitely many derivatives. The function $F : V \to V$ $F(f) = 3f' - 2f''$ is linear.
3. If the determinant of a 4×4 matrix is 4, then the rank of the matrix must be 4.
4. If the standard vectors $\{e_1, e_2, \ldots e_n\}$ are eigenvectors of an $n \times n$ matrix, then the matrix is diagonal.
5. If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be I_n.
6. If two 3×3 matrices both have the eigenvalues $3, 4, 5$, then A must be similar to B.

Problem 8 Given an operator T that has in some basis a matrix $M(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, prove that there exists no basis , in which T has a diagonal matrix . (Do not simply quote facts about Jordan Canonical Form but give a direct proof.)

Problem 9 Let V be a finite dimensional vector space and T, S linear transformations which commute, i.e. $TS = ST$ and T and S are both diagonalizable, show that T and S are simultaneous diagonalizable, that is there exists a common basis of eigenvectors for both T and S

Problem 10 Let M be a real diagonalizable $n \times n$ matrix. Prove that there is an $n \times n$ matrix N with real entries such that $N^3 = M$.

Problem 11 Let $\lambda_1, \lambda_2, ..., \lambda_n$ be distinct elements of the field $\mathbb{F}$. Then the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is invertible. (Use the fact that a nonzero polynomial of degree less then n can not have n roots.)

Problem 12 Find the eigenvalues of the matrix A, given below. Find bases for the eigenspaces of A. Can you find an invertible matrix, S, such that $S^{-1}AS = D$, where D is a diagonal matrix? If no, why not? If yes, find the matrices S and D.

1.
$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 6 & 6 & -5 \end{bmatrix}$$

2.
$$A = \begin{bmatrix} -8 & 5 & 4 \\ -9 & 5 & 5 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 13 1. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 & 0 \\ 5 & 2 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

2. What is the common denominator of the entries in A^{-1}.

Problem 14 A two by two matrix A has a trace $trA = 8$ and determinant $detA = 12$. Is A digonalizable?

Problem 15 A two by two matrix A has a characteristic polynomial $7 - 8t + t^2$. In addition

$$A^2 = \begin{bmatrix} 41 & -40 \\ -8 & 9 \end{bmatrix}$$

Find A.

Problem 16 Find the general solution of $y^{(4)} - 8y^{(2)} + 16y = 0$.