MAT310 Fall 2012

Practice Final

The actual Final exam will consist of twelve problems that cover chapters 1.2-5.4

(inclusive) with omission of 2.6,4.5,5.3
Problem 1 Let $V, W$ be vector spaces. Define the following terms:

1. What is a subspace of $V$?

2. Let $F : V \rightarrow W$ be a function. What does it mean to say that $F$ is linear?

3. Let $T = \{v_1, v_2, \ldots\}$ be a subset of $V$. What is a linear combination of elements of $T$? What is the span of $T$? What does it mean to say that $T$ is linearly independent? What does it mean to say that $T$ spans $V$? What does it mean to say that $T$ is a basis of $V$?

4. What is the dimension of $V$?

5. Let $F : V \rightarrow W$ be linear. Define $N(F) = \ker(F)$. Define $\text{im}(F)$. What is the rank of $F$? What is the nullity of $F$?

6. Let $F : V \rightarrow V$ be linear. What is an eigenvalue of $F$? What is an eigenvector of $F$?

7. What does it means to say that two $n \times n$ matrices are similar?

8. What does it mean to say that two vector spaces are isomorphic?

9. Let $A$ be an $n \times n$ matrix. What is an eigenbasis for the matrix $A$?

10. Let $B$ be a basis of a vector space $V$. What does one mean by the coordinates of a vector $v \in V$ with respect to $B$?
**Problem 2**

1. Let $F : V \to W$ be linear. Show that $\ker(F)$ is a subspace of $V$.

   Show that $\text{im}(F)$ is subspace of $W$.

2. State the rank-nullity theorem.
Problem 3 Consider the system of equations:

\[
\begin{align*}
    x - 2y + 3z - w &= 2 \\
    2x + y - z + 3w &= 1 \\
    5x + z + 5w &= 4
\end{align*}
\]

1. Find all, if any, solutions to this system.

2. Write the system as a matrix equation.
**Problem 4** Determine linearly independent sets

1. Set of functions 1, $e^x$ and $e^{2x}$ thought of as elements of real linear space of continuous functions $C[0, 1]$

2. Set of functions 1, $\sin^2(x)$ and $3 - \cos^2(x)$ thought of as elements of real linear space of continuous functions $C[0, 2\pi]$
Problem 5 Let $P : V \to V$ be a projection on a finite dimensional vector space, i.e., $P$ is a linear map with the property that $P^2 = P$. Show that there exists a basis $B$ for $V$ such that $M(P) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where $I_r$ is the $r \times r$ identity matrix. (Here $M(P)$ is the matrix representing the map $P$ relative to the basis $B$.)
**Problem 6** Find bases in $\text{Im}A$ and $\text{ker}A$ where the linear transformation $A : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ has a matrix
\[
\begin{bmatrix}
1 & -2 & 1 & 2 & 0 \\
2 & 1 & 1 & -1 & 2 \\
5 & 0 & 3 & 0 & 4
\end{bmatrix}
\]
Extend the bases to bases of $\mathbb{R}^3, \mathbb{R}^5$ respectively.
Problem 7 True or False. (Explain!)

1. The set of all vectors of the form \((a, b, 0, b)^t\) where \(a, b\) are real numbers forms a subspace in \(\mathbb{R}^4\).

2. Let \(V\) be the space of all functions from \(\mathbb{R}\) to \(\mathbb{R}\) that have infinitely many derivatives. The function \(F : V \rightarrow V F(f) = 3f' - 2f''\) is linear.

3. If the determinant of a \(4 \times 4\) matrix is 4, then the rank of the matrix must be 4.

4. If the standard vectors \(\{e_1, e_2, \ldots, e_n\}\) are eigenvectors of an \(n \times n\) matrix, then the matrix is diagonal.

5. If 1 is the only eigenvalue of an \(n \times n\) matrix \(A\), then \(A\) must be \(I_n\).

6. If two \(3 \times 3\) matrices both have the eigenvalues 3, 4, 5, then \(A\) must be similar to \(B\).
Problem 8 Given an operator \( T \) that has in some basis a matrix \( M(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), prove that there exists no basis, in which \( T \) has a diagonal matrix. (Do not simply quote facts about Jordan Canonical Form but give a direct proof.)
**Problem 9** Let $V$ be a finite dimensional vector space and $T, S$ linear transformations which commute, i.e. $TS = ST$ and $T$ and $S$ are both diagonalizable, show that $T$ and $S$ are simultaneous diagonalizable, that is there exists a common basis of eigenvectors for both $T$ and $S$.
Problem 10 Let $M$ be a real diagonalizable $n \times n$ matrix. Prove that there is an $n \times n$ matrix $N$ with real entries such that $N^3 = M$. 
Problem 11 Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct elements of the field $F$. Then the matrix

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\cdots & \cdots & \cdots & \cdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{pmatrix}
$$

is invertible. (Use the fact that a nonzero polynomial of degree less than $n$ can not have $n$ roots.)
Problem 12 Find the eigenvalues of the matrix $A$, given below. Find bases for the eigenspaces of $A$. Can you find an invertible matrix, $S$, such that $S^{-1}AS = D$, where $D$ is a diagonal matrix? If no, why not? If yes, find the matrices $S$ and $D$.

1. 

$$
A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 6 & 6 & -5 \end{bmatrix}
$$

2. 

$$
A = \begin{bmatrix} -8 & 5 & 4 \\ -9 & 5 & 5 \\ 0 & 1 & 0 \end{bmatrix}
$$
Problem 13

1. Find the determinant of the matrix

\[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 \\
0 & 5 & 0 & 3 & 0 \\
0 & 3 & 0 & 2 & 0 \\
5 & 2 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & 2
\end{bmatrix}
\]

2. What is the common denominator of the entries in \( A^{-1} \).
Problem 14  A two by two matrix $A$ has a trace $\text{tr}A = 8$ and determinant $\text{det}A = 12$. Is $A$ diagonalizable?
Problem 15 A two by two matrix $A$ has a characteristic polynomial $7 - 8t + t^2$. In addition

$$A^2 = \begin{bmatrix} 41 & -40 \\ -8 & 9 \end{bmatrix}$$

Find $A$. 
Problem 16  Find the general solution of $y^{(4)} - 8y^{(2)} + 16y = 0$. 