Problem 1  Let $F = \mathbb{R}^2, U = \{(x, 0) \mid x \in \mathbb{R}\}, W = \{(0, y) \mid y \in \mathbb{R}\}$. Then $(1, 1) = (1, 0) + (0, 1)$ is not in $U \cup W$ but it should have been if $U \cup W$ was a subspace.

If $U \cup W$ is a subspace of $F$ and $U \not\subseteq W, W \not\subseteq U$, choose $u \in U \setminus W$ and $w \in W \setminus U$. Then $u + w \in U \cup W$, since it is a subspace. If $u + w \in U$ then $w = (u + w) - u \in U$, a contradiction. On the other hand, if $u + w \in W$ then $u = (u + w) - w \in W$, again a contradiction.

Problem 2  (i) Suppose $a + b(t - 1) + c(t - 1)^2 + d(t - 1)^3 = 0$. Then

$$
(a - b + c - d) + (b - 2c - 3d)t + (c + 3d)t^2 + dt^3 = 0.
$$

Since $(1, t, t^2, t^3)$ is a basis of $\mathbb{P}_3$ we conclude that $d = 0$. Since $c + 3d = 0$ this forces $c = 0$. Now $b - 2c - 3d = 0$ implies $b = 0$ and $a - b + c - d = 0$ implies $a = 0$. This proves that $(1, t - 1, (t - 1)^2, (t - 1)^3)$ is linearly independent. Since $\mathbb{P}_3$ has dimension 4 and $U = \text{span}(1, t - 1, (t - 1)^2, (t - 1)^3)$ is a subspace of dimension 4, we conclude that $U = \mathbb{P}_3$.

(ii) Yes. For example, take $S = \{(1, 0), (0, 1)\}$ and $T = \{(1, 1), (1, -1)\}$. The vectors in $S$ span $\mathbb{R}^2$ and so does the vectors in $T$ but $S \neq T$.

Problem 3  It is given that $\psi \phi : V \rightarrow V$ is an isomorphism, i.e., it is injective (and surjective as well). If $\phi(v) = 0$ then $\psi \phi(v) = 0$ whence $v = 0$. Therefore, $\phi$ is injective. On the other hand, given $v \in V$, let $v' \in V$ be the unique element such that $\psi \phi(v') = v$. This is possible since $\psi \phi$ is surjective. Then $\psi(\phi(v')) = v$ and $\phi(v') \in W$, whence $\psi$ is surjective.

Problem 4  Let $\rho : V \rightarrow V$ be such that $\rho \rho = \rho$. Let $v \in \text{range}(\rho)$ and write $v = \rho(v')$ for some $v' \in V$. Then

$$
\rho(v) = \rho \rho(v') = \rho(v') = v.
$$

Thus, $\rho$ is the identity on range($\rho$).

Problem 5  It is enough to prove linear independence since then the span of the given vectors would be of dimension 3 and consequently has to be $\mathbb{R}^3$. Suppose

$$
a(1, 1, 0) + b(2, 0, -1) + c(-3, 1, 1) = (a + 2b - 3c, a + c, -b + c) = (0, 0, 0).
$$

This implies that $b = c, a = -c$ and $a + 2b - 3c = 0$. The last equation can be written as $-c + 2c - 3c = 0$ whence $c = 0$ and $a = b = 0$.

Problem 6  Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $\phi(x, y) = (0, x)$. Since $\phi \phi(x, y) = \phi(0, x) = (0, 0)$ it defines a nilpotent endomorphism of order 2. Similarly, $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\psi(x, y) = (y, 0)$ is also a nilpotent endomorphism of order 2. Now $\psi \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given
by $\psi\phi(x, y) = \psi(0, x) = (x, 0)$. It is clear that $(\psi\phi)^2(x, y) = \psi\phi(x, 0) = (x, 0) = \psi\phi(x, y)$. Therefore, $\psi\phi$ is an idempotent.

**Problem 7** Since $x \in \text{span}\{M, y\}$ and $x \not\in M$ we can write $x = a_1v_1 + \cdots + a_kv_k + b$ where $v_i$'s are a basis of $M$ and $b \neq 0$. Then $y = (-a_1/b)v_1 + \cdots + (-a_k/b)v_k + (1/b)x$ and $y \in \text{span}\{M, x\}$. Clearly $M \subset \text{span}\{M, x\}$. Therefore, $\text{span}\{M, y\} \subset \text{span}\{M, x\}$. On the other hand, $x \in \text{span}\{M, y\}$ whence $\text{span}\{M, x\} \subset \text{span}\{M, y\}$. This proves that $\text{span}\{M, y\} = \text{span}\{M, x\}$.

**Problem 8** Since $M \subset M + (L \cap N)$ this implies that $L \cap M \subset L \cap (M + (L \cap N))$. On the other hand $L \cap N = L \cap (L \cap N) \subset L \cap (M + (L \cap N))$. This means that $L \cap M$ and $L \cap N$ are both subspaces of $L \cap (M + (L \cap N))$ and therefore contains the sum as well, viz., $(L \cap M) + (L \cap N) \subset L \cap (M + (L \cap N))$.

On the other hand if $v \in L \cap (M + (L \cap N))$ then $v \in L$ and $v \in M + (L \cap N)$. Write $v = m + l$ where $m \in M$ and $l \in L \cap N$. Then $m = v - l \in L$ whence $m \in L \cap M$. Therefore, $v = m + l \in (L \cap M) + (L \cap N)$.

**Problem 9** (i) If $(1, \alpha) = \lambda(1, \beta)$ then $\lambda = 1$ and $\alpha = \beta$. Therefore, $(1, \alpha)$ and $(1, \beta)$ are linearly independent if and only if $\alpha \neq \beta$.

(ii) No. If there were then $\mathbb{C}^2$ would contain the span of these three vectors which is a 3 dimensional subspace while $\mathbb{C}^2$ is only 2 dimensional.

(iii) No matter what $x \in \mathbb{C}$ is, the vectors $(1, 1, 1)$ and $(1, x, x^2)$ span a subspace of $\mathbb{C}^3$ of dimension at most 2. When $x = 1$ the span is $\{(z, z, z) \mid z \in \mathbb{C}\}$. When $x \neq 1$ the span is a 2 dimensional subspace. In either case, it does not span $\mathbb{C}^3$.

(iv) If these vectors are linearly independent then we’ll be done since we’re in $\mathbb{C}^3$. For any choice of $x \in \mathbb{C}$ we can write $(x, 1, 1 + x) = (x, 0, 1) + (0, 1, x)$ whence they are not linearly independent and therefore not a basis.

**Problem 10** (i) The first and the third transformations are linear. The second is not since $T(2x, 2y) = 4T(x, y)$.

(ii) The first and the third are linear transformations. For example, in the first case $T(a_0 + a_1x + \cdots + a_kx^k) = a_0 + a_1x^2 + \cdots + a_kx^{2k} = T(a_0) + a_1T(x) + a_2T(x^2) + \cdots + a_kT(x^k)$ which precisely means that $T$ is linear. Similarly, in the third case $T(a_0 + a_1x + \cdots + a_kx^k) = X^2(a_0 + a_1x + \cdots + a_kx^k) = T(a_0) + a_1T(x) + a_2T(x^2) + \cdots + a_kT(x^k)$ which implies linearity of $T$. In the second case, however, $T(2p(x)) = 4(p(x))^2 \neq 2T(p(x))$ whence $T$ is not linear.
Problem 11  (i) Let \( p(x) = a_0 + a_1x + \cdots + a_6x^6 \in \mathcal{P}_6 \).

\[
T(p(x)) := \int_{-3}^{x+9} p(t) \, dt = \sum_{i=0}^{6} a_i \int_{-3}^{x+9} t^i \, dt = \sum_{i=0}^{6} \frac{a_i}{i+1} ((x+9)^{i+1} - (-3)^{i+1}).
\]

If \( T(p(x)) = 0 \) then \( a_6 \), the coefficient of \( x^7 \), is zero. Therefore,

\[
T(p(x)) = \sum_{i=0}^{5} \frac{a_i}{i+1} ((x+9)^{i+1} - (-3)^{i+1}) = 0.
\]

Again, the coefficient of \( x^6 \) is \( a_5 \) and it has to be zero. Doing this recursively leads one to \( T(p(x)) = a_0((x+9) - (-3)) = a_0(x+12) = 0 \) whence \( a_0 = 0 \). Therefore, if \( p(x) \in \text{null}(T) \) then \( p(x) = 0 \). So \( \text{null}(T) = \{0\} \).

(ii) Let \( p(x) = a_0 + a_1x + \cdots + a_5x^5 \in \mathcal{P}_5 \) such that

\[
0 = D(p(x)) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4.
\]

Then \( a_1 = a_2 = a_3 = a_4 = a_5 = 0 \). Therefore, \( \text{null}(D) = \mathbb{R} \), the space of constant polynomials.

(iii) If \( T(x,y) = 0 \) then \( 2x + 3y = 0 \) and \( 7x = 5y \). Combining both these we get \( -2x/3 = 7x/5 \) which means \( x = 0 \) and \( y = 7x/5 = 0 \). Therefore, \( \text{null}(T) = \{0\} \).

(iv) We know that \( (1,x,x^2,x^3,x^4,x^5) \) is a basis for \( \mathcal{P}_5 \). It follows from the definition of \( T \) that \( T(x^i) = x^{4i} \neq 0 \), i.e., \( T \) is injective on the basis elements and therefore injective on \( \mathcal{P}_5 \). Consequently, \( \text{null}(T) = \{0\} \).

(v) If \( T(x,y) = (x,0) = (0,0) \) then \( x = 0 \). Therefore, \( \text{null}(T) = \{(0,y) \mid y \in \mathbb{R}\} \).

(vi) If \( T(x,y) = x+2y = 0 \) then \( y = -x/2 \). Therefore, \( \text{null}(T) = \{(2x,-x) \mid x \in \mathbb{R}\} \).

Problem 12  (i) We compute \( ST \) and \( TS \) and then compare them. On the one hand

\[
ST(p(x)) = S(x^2p(x)) = x^4p(x^2)
\]

while on the other hand

\[
TS(p(x)) = T(p(x^2)) = x^2p(x^2).
\]

Therefore \( S \) and \( T \) don’t commute.

(ii) As before, on the one hand

\[
ST(a + bx + cx^2 + dx^3) = S(a + cx^2) = a + c(x+2)^2 = (a+4c) + 2cx + cx^2
\]

while on the other hand

\[
TS(a + bx + cx^2 + dx^3) = T(a + 2b + 4c + 8d + (b+2c+12d)x + (c+6d)x^2 + dx^3)
\]

\[
= a + 2b + 4c + 8d + (c+6d)x^2.
\]

Therefore, \( S \) and \( T \) don’t commute.

Problem 13  (i) No. Any invertible linear transformation must be surjective, viz., the image must have full dimension. In this case, the image of \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is \( \{(x,x) \mid x \in \mathbb{R}\} \) is 1 dimensional.

(ii) Yes. The inverse of \( T \) is \( T \) itself. For example, \( TT(x,y) = T(y,x) = (x,y) \) whence \( TT = \text{Id} \).

(iii) No. Any invertible linear transformation must be injective, viz., it must have no null space. As we saw in 11 (ii), \( D \) on \( \mathcal{P}_5 \) has a 1 dimensional space as its null space and hence not invertible.
Problem 4.

1. Let \( \{u_1, \ldots, u_k\} \) be a basis of \( L \), where \( k = \dim L \). For any \( \varphi(u) \in \varphi(L) \), \( \mu \in L \), write \( \mu = a_1 u_1 + \cdots + a_k u_k \). Then

\[
\varphi(u) = \varphi(a_1 u_1 + \cdots + a_k u_k) = a_1 \varphi(u_1) + \cdots + a_k \varphi(u_k)
\]

That is, \( \varphi(u_1), \ldots, \varphi(u_k) \) span \( \varphi(L) \). Therefore, \( \dim \varphi(L) \leq k = \dim L \).

2. When \( \varphi \) is one to one, if \( a_1 \varphi(u_1) + \cdots + a_k \varphi(u_k) = 0 \), which is equivalent to

\[
\varphi(a_1 u_1 + \cdots + a_k u_k) = 0,
\]

then \( a_1 = \cdots = a_k = 0 \). (\( \text{Null}(\varphi) = \{0\} \))

Since \( \{u_1, \ldots, u_k\} \) is a basis of \( L \), we must have \( a_1 = \cdots = a_k = 0 \). Therefore \( \varphi(u_1), \ldots, \varphi(u_k) \) are linearly independent.

Combining (1), we have \( \{\varphi(u_1), \ldots, \varphi(u_k)\} \) is a basis of \( \varphi(L) \), so \( \dim \varphi(L) = k = \dim L \).

Problem 5.

\( \{1, x, x^2, x^3\} \) is a basis of \( P_3 \). Denote it by \( \beta \).

Denote \( \gamma \) the basis \( \{(1,0), (0,1)\} \) of \( \mathbb{R}^2 \).

Then \( [T]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \)

\( \text{rank} \left( [T]_\gamma^\beta \right) = 2 \quad \Rightarrow \quad \dim N(T) = 2 \)

Actually \( \left\{ x^3 - x^2, x^2 - x \right\} \) is a basis of \( N(T) \), denote it by \( \delta \).
Define \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by \( A(f) = (f', f'') \),
then \( [A]_\gamma^\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \), \( \gamma \) as defined above.

\[ \det ([A]_\gamma^\alpha) \neq 0 \Rightarrow A \text{ isomorphism}. \]

\( \alpha \) can be extended to \( f, x^2, x, 1 \), a basis of \( \mathbb{R}_3 \).

**Problem 16.**

\[
\begin{align*}
T(1, 3) &= (-7, 26) = -54(1, 3) + 47(1, 4) \\
T(1, 4) &= (-10, 33) = -76(1, 3) + 66(1, 4)
\end{align*}
\]

\[ \Rightarrow [T]_\beta^\alpha = \begin{pmatrix} -54 & -76 \\ 47 & 66 \end{pmatrix} \]

\[
\begin{align*}
T(3, 2) &= (0, 29) = -203(3, 2) + 87(7, 5) \\
T(7, 5) &= (-1, 70) = -495(3, 2) + 212(7, 5)
\end{align*}
\]

\[ \Rightarrow [T]_\beta^\alpha' = \begin{pmatrix} -203 & -495 \\ 87 & 212 \end{pmatrix} \]

\[ Q = [T]_\beta^\alpha' = \begin{pmatrix} -16 & -23 \\ 7 & 10 \end{pmatrix} \]

\[ [T]_\beta^\alpha, Q = \begin{pmatrix} -217 & -281 \\ 92 & 119 \end{pmatrix} = Q [T]_\beta^\alpha \]