Problem 2

i. $xy' + y = 3$

This is of the form $a(x)y' + b(x)y = f(x)$, so it is linear. No integrating factor is needed (or you may use $p(x) = 1$).

$$
xy' + y = 3 \\
(xy)' = 3 \\
xy = 3x + c \\
y = 3 + \frac{c}{x}
$$

ii. $xy' - y = 2x^2$

This is linear, and $xy' - y = 2x^2 \Rightarrow y' - \frac{y}{x} = 2x$, so the integrating factor is given by $p(x) = e^{\int -\frac{1}{x} \, dx} = \frac{1}{x}$.

$$
\frac{1}{x} \left( y' - \frac{y}{x} \right) = \frac{1}{x} \cdot 2x \\
y' - \frac{y}{x} = 2 \\
\left( \frac{y}{x} \right)' = 2 \\
\frac{y}{x} = 2x + c \\
y = 2x^2 + cx
$$

iii. $y' - \frac{3}{x-1}y = (x - 1)^4$

This is linear and has integrating factor $p(x) = e^{\int -\frac{3}{x-1} \, dx} = (x - 1)^{-3}$.
\[
(x - 1)^{-3} \left( y' - \frac{3}{x - 1} y \right) = (x - 1)^{-3} (x - 1)^4 \\
(x - 1)^{-3} y' - \frac{3}{(x - 1)^4} y = (x - 1) \\
((x - 1)^{-3} y)' = x - 1 \\
(x - 1)^{-3} y = \frac{1}{2} x^2 - x + c \\
y = (x - 1)^3 \left( \frac{1}{2} x^2 - x + c \right)
\]

iv. \( y' + \frac{1}{\sin x} y - y^2 = 0 \)

This is not linear because of the \( y^2 \) term.

v. \( xy' + y = x^5 \)

This is linear with integrating factor \( p(x) = 1 \).

\[
xy' + y = x^5 \\
(xy)' = x^5 \\
xy = \frac{1}{6} x^6 + c \\
y = \frac{1}{6} x^5 + \frac{c}{x}
\]

Problem 4

i. \( \frac{dy}{dx} = (x + y)/(2x - y) \)

\[
\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{2 - \frac{y}{x}}
\]

Make the substitution \( u = \frac{y}{x} \). Then \( \frac{dy}{dx} = x \frac{du}{dx} + u \) and so

\[
x \frac{du}{dx} + u = \frac{1 + u}{2 - u} \\
x \frac{du}{dx} = \frac{u^2 - u + 1}{2 - u} \\
- \frac{u - 2}{u^2 - u + 1} du = \frac{1}{x} dx
\]
To compute the antiderivative \( \int \frac{u-2}{u^2-u+1} \, du \) make the substitution \( w = u^2 - u + 1 \). Then \( dw = (2u - 1) \, du \) and the integral becomes

\[
\int \frac{u-2}{u^2-u+1} \, du = \frac{1}{2} \int \frac{dw}{w} - \frac{3}{2} \int \frac{du}{u^2-u+1}.
\]

The first integral on the right is

\[
\int \frac{dw}{w} = \ln |w| + c_1,
\]

and for the second we complete the square in the denominator to obtain

\[
\int \frac{du}{u^2-u+1} = \int \frac{du}{(u-\frac{1}{2})^2 + \frac{3}{4}} = \frac{4}{3} \int \frac{du}{\left( \frac{2}{\sqrt{3}} u - \frac{1}{\sqrt{3}} \right)^2 + 1} = \frac{2}{\sqrt{3}} \arctan \left( \frac{2}{\sqrt{3}} u - \frac{1}{\sqrt{3}} \right) + c_2.
\]

So we have

\[
\frac{2}{\sqrt{3}} \arctan \left( \frac{3}{\sqrt{3}} u - \frac{1}{\sqrt{3}} \right) - \frac{1}{2} \ln |w| = \ln |x| + c,
\]

where \( u = \frac{y}{x} \) and \( w = (\frac{y}{x})^2 - \frac{y}{x} + 1 \).

\[\text{ii. } dy/dx = xy + xy^4\]

\[
\frac{dy}{dx} = x(y + y^4)
\]

\[
\frac{dy}{y + y^4} = x \, dx
\]

\[
\frac{dy}{y(1 + y^3)} = x \, dx
\]

Use partial fractions to express \( \frac{1}{y(1+y^3)} \) in the form

\[
\frac{1}{y(1+y^3)} = \frac{A}{y} + \frac{By^2 + Cy + D}{1+y^3}
\]

We find that \( A = 1, B = -1, C = 0, \) and \( D = 0 \).

\[
\frac{1}{y(1+y^3)} = \frac{1}{y} - \frac{y^2}{1+y^3}
\]

This means we can write
\[
\left( \frac{1}{y} - \frac{y^2}{1 + y^3} \right) dy = x dx
\]
\[
\ln |y| - \ln |1 + y^3| = \frac{1}{2} x^2 + c
\]
\[
\ln \left| \frac{y}{1 + y^3} \right| = \frac{1}{2} x^2 + c
\]
\[
\frac{y}{1 + y^3} = ae^{\frac{1}{2} x^2}.
\]

**Problem 7**

i. \(y'' - y' - 2y = t^2 e^{2t}, \ y(0) = 0, \ y'(0) = 1\)

First find solutions to the homogeneous equation \(y'' - y' - 2y = 0\). The associated characteristic polynomial is

\[
r^2 - r - 2 = (r - 2)(r + 1).
\]

Thus we have two independent complementary solutions \(y_1 = e^{2t}\) and \(y_2 = e^{-t}\). Now we write the particular solution as

\[
y_p = u_1(t)y_1(t) + u_2(t)y_2(t),
\]

for some functions \(u_1\) and \(u_2\). We determine \(u_1\) and \(u_2\) by solving the system

\[
\begin{align*}
    u_1'y_1 + u_2'y_2 &= 0 \\
    u_1'y_1' + u_2'y_2' &= t^2 e^{2t}
\end{align*}
\]

This is

\[
\begin{align*}
    u_1'e^{2t} + u_2'e^{-t} &= 0 \\
    2u_1'e^{2t} - u_2'e^{-t} &= t^2 e^{2t}
\end{align*}
\]

We find

\[
    u_1' = \frac{t^2}{3} \quad \text{and} \quad u_2' = -\frac{t^2}{3} e^{3t}.
\]

Thus we may take

\[
    u_1 = t^3 \quad \text{and} \quad u_2 = -\left( \frac{1}{9} t^2 - \frac{2}{27} t + \frac{2}{81} \right) e^{3t}
\]

Thus

\[
y_p = t^3 e^{2t} - \left( \frac{1}{9} t^2 - \frac{2}{27} t + \frac{2}{81} \right) e^{2t}.
\]
We may drop the $\frac{2}{81}e^{2t}$ term since it appears in the homogeneous solution. The general solution $y = y_p + c_1y_1 + c_2y_2$ in this case may be written

$$y = \left( t^3 - \frac{1}{9}t^2 + \frac{2}{27}t \right)e^{2t} + c_1 e^{2t} + c_2 e^{-t}. $$

Initial conditions give $y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$. $y'(0) = 1 \Rightarrow \frac{2}{27} + 2c_1 - c_2 = 1 \Rightarrow c_1 = \frac{25}{81}$. So the final answer is

$$y = \left( t^3 - \frac{1}{9}t^2 + \frac{2}{27}t \right)e^{2t} + \frac{25}{81}e^{2t} - \frac{25}{81}e^{-t}. $$

**ii.** $y'' + y = -2\sin t$, $y(0) = 1$, $y'(0) = 1$

The characteristic equation is $r^2 + 1 = 0$. This gives homogeneous solutions $y_1 = \cos t$ and $y_2 = \sin t$. We solve the system

$$u_1' \cos t + u_2' \sin t = 0$$
$$-u_1' \sin t + u_2' \cos t = -2 \sin t$$

which gives $u_1' = 2\sin^2 t$ and $u_2' = -2 \sin t \cos t$. Using the identity $2 \sin^2 t = 1 - \cos 2t$, we find

$$u_1 = t - \frac{1}{2} \sin 2t \quad \text{and} \quad u_2 = \cos^2 t.$$ 

The particular solution is thus

$$y_p = \left( t - \frac{1}{2} \sin 2t \right) \cos t + \cos^2 t \sin t$$
$$= (t - \sin t \cos t) \cos t + (1 - \sin^2 t) \sin t$$
$$= t \cos t - \sin t \cos^2 t + \sin t - \sin^3 t$$
$$= t \cos t - \sin t (1 - \sin^2 t) + \sin t - \sin^3 t$$
$$= t \cos t$$

so the general solution is

$$y = t \cos t + c_1 \cos t + c_2 \sin t.$$ 

The initial conditions show $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 1 \Rightarrow c_2 = 0$, so the final answer is

$$y = t \cos t + \cos t = (t + 1) \cos t.$$ 

**Problem 9**

$$y'' + 5y' + 6y = 0$$
i. The characteristic equation is

\[ r^2 + 5r + 6 = 0 \Rightarrow (r + 2)(r + 3) = 0. \]

So the general solution is

\[ y = c_1 e^{-2t} + c_2 e^{-3t}. \]

ii. Put \( x = y' \). Then \( x' = y'' \), so we obtain the first order system

\[
\begin{cases}
  x' = -5x - 6y \\
y' = x
\end{cases}
\]

which may be expressed using matrices in the form \( \vec{x}' = P \vec{x} \)

\[
\begin{bmatrix}
  x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
  -5 & -6 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
y
\end{bmatrix}
\]

We find the eigenvalues by equating the determinant of \( \lambda I - P \) to 0

\[
\begin{vmatrix}
  -5 - \lambda & -6 \\
  1 & -\lambda
\end{vmatrix} = \lambda(\lambda + 5) + 6 = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2)
\]

this has solutions \( \lambda = -3 \) and \( \lambda = -2 \). Substituting these into \( (\lambda I - P)(\vec{v}) = 0 \) we find solutions

\[
\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

This gives

\[
\begin{bmatrix}
  x \\
y
\end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

**Problem 12**

Without damping we have the equation \( F = -ku \). Given that it takes 8lbs to stretch the spring \( \frac{1}{2} \) ft, we find that \( -8 = -k \frac{1}{2} \), so \( k = 16 \text{ft-lb} \). We also know that the acceleration due to gravity is \( 32 \text{ft/s}^2 \). Using \( F = ma \) where \( F = 8 \) is the weight of the object, we find its mass to be \( m = 8/32 = 1/4 \) slugs.

i. The motion of the mass is given by

\[ \frac{1}{4} u'' + 2u' + 16u = \cos 3t, \quad u(0) = \frac{1}{6}, \quad u'(0) = 0 \]

where \( u(t) \) denotes the displacement from equilibrium (its natural dangling position) \( t \) seconds after it is released. We have \( u'(0) = 0 \) since it is released from rest.

ii. The characteristic equation is \( \frac{1}{4} r^2 + 2r + 16 = 0 \) which has roots \( -4 \pm 4\sqrt{3}i \). Therefore the complimentary solution is \( u_c = e^{-4t} \left[ c_1 \cos(4\sqrt{3}t) + c_2 \sin(4\sqrt{3}t) \right] \).
Using the method of undetermined coefficients, we guess a particular solution to have the form

\[ u_p = A \cos 3t + B \sin 3t. \]

Substituting this into the equation of motion gives

\[-\frac{9}{4} A \cos 3t - \frac{9}{4} B \sin 3t - 6A \sin 3t + 6B \cos 3t + 16A \cos 3t + 16B \sin 3t = \cos 3t.\]

This is equivalent to the system of equations

\[\begin{align*}
-\frac{9}{4} A + 6B + 16A &= 1 \\
-\frac{9}{4} B - 6A + 16B &= 0
\end{align*}\]

we find \( A = \frac{220}{3601} \) and \( B = \frac{96}{3601} \). The general solution is given by

\[ u = e^{-4t} \left[ c_1 \cos(4\sqrt{3}t) + c_2 \sin(4\sqrt{3}t) \right] + A \cos 3t + B \sin 3t. \]

The initial conditions \( u(0) = \frac{1}{6} \) and \( u'(0) = 0 \) yield the system

\[\begin{align*}
A + c_1 &= 1/6 \\
-4c_1 + 4\sqrt{3}c_2 + 3B &= 0
\end{align*}\]

Solving gives \( c_1 = \frac{2281}{31006} \) and \( c_2 = \frac{1849}{64818} \sqrt{3}. \)

**Problem 13**

The operational determinant is given by

\[ L = \begin{vmatrix} D^2 + 1 & -D^2 \\ D^2 & D^2 - 1 \end{vmatrix} = (D^2 + 1)D^2 - D^2(D^2 - 1) = 2D^4 \]

This gives the equations

\[ 2D^4 x = \begin{vmatrix} 2e^{-t} & 0 \\ -D^2 & D^2 \end{vmatrix} = 2e^{-t} \quad \text{and} \quad 2D^4 y = \begin{vmatrix} D^2 + 1 & 2e^{-t} \\ D^2 & 0 \end{vmatrix} = 0 \]

\( D^4 x = e^{-t} \) has the general solution \( x = a_3t^3 + a_2t^2 + a_1t + a_0 + e^{-t} \) and \( D^4 y = 0 \) has the general solution \( y = b_3t^3 + b_2t^2 + b_1t + b_0. \) Resubstituting these back into the second equation \((D^2 - 1)x + D^2y = 0\) yields

\[ 6a_3t + 2a_2 - a_3t^3 - a_2t^2 - a_1t - a_0 + 6b_3t + 2b_2 = 0.\]

Thus matching like terms gives
\[ \begin{align*}
2a_2 - a_0 + 2b_2 &= 0 \\
6a_3 - a_1 + 6b_3 &= 0 \\
\quad a_2 &= 0 \\
\quad a_3 &= 0
\end{align*} \]

We obtain the same system if we substitute into the first equation \((D^2 + 1)x - D^2y = 2e^{-t}\). Thus the general solution is

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 + a_1 t + e^{-t} \\ b_0 + b_1 t + \frac{a_0}{2} t^2 + \frac{a_1}{6} t^3 \end{pmatrix}
\]

where \(a_0, a_1, b_0,\) and \(b_1\) are arbitrary constants.