MAT303 Spring 2009

Some Practice Final Solutions

Problem 2

i. $xy' + y = 3$

This is of the form $a(x)y' + b(x)y = f(x)$, so it is linear. No integrating factor is needed (or you may use $p(x) = 1$).

$$\begin{aligned} xy' + y &= 3 \\ (xy)' &= 3 \\ xy &= 3x + c \\ y &= 3 + \frac{c}{x} \end{aligned}$$

ii. $xy' - y = 2x^2$

This is linear, and $xy' - y = 2x^2 \Rightarrow y' - \frac{y}{x} = 2x$, so the integrating factor is given by $p(x) = \mathrm{e}^{\int -\frac{1}{x}\,dx} = \frac{1}{x}$.

$$\begin{aligned} \frac{1}{x}\left(y' - \frac{y}{x}\right) &= \frac{1}{x}2x \\ \frac{y'}{x} - \frac{y}{x^2} &= 2 \\ \left(\frac{y}{x}\right)' &= 2 \\ \frac{y}{x} &= 2x + c \\ y &= 2x^2 + cx \end{aligned}$$

iii. $y' - \frac{3}{x-1}y = (x-1)^4$

This is linear and has integrating factor $p(x) = \mathrm{e}^{\int -\frac{3}{x-1}\,dx} = (x-1)^{-3}$.

$$\begin{aligned}
(x-1)^{-3}\left(y' - \frac{3}{x-1}y\right) &= (x-1)^{-3}(x-1)^4 \\
(x-1)^{-3}y' - \frac{3}{(x-1)^4}y &= (x-1) \\
\left((x-1)^{-3}y\right)' &= x-1 \\
(x-1)^{-3}y &= \frac{1}{2}x^2 - x + c \\
y &= (x-1)^3\left(\frac{1}{2}x^2 - x + c\right)
\end{aligned}$$

iv. $y' + \frac{1}{\sin x}y - y^2 = 0$

This is not linear because of the y^2 term.

v. $xy' + y = x^5$

This is linear with integrating factor $p(x) = 1$.

$$\begin{aligned}
xy' + y &= x^5 \\
(xy)' &= x^5 \\
xy &= \frac{1}{6}x^6 + c \\
y &= \frac{1}{6}x^5 + \frac{c}{x}
\end{aligned}$$

Problem 4

i. $dy/dx = (x+y)/(2x-y)$

$$\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{2 - \frac{y}{x}}$$

Make the substitution $u = \frac{y}{x}$. Then $\frac{dy}{dx} = x\frac{du}{dx} + u$ and so

$$\begin{aligned}
x\frac{du}{dx} + u &= \frac{1+u}{2-u} \\
x\frac{du}{dx} &= \frac{u^2 - u + 1}{2-u} \\
-\frac{u-2}{u^2 - u + 1}\,du &= \frac{1}{x}\,dx
\end{aligned}$$

To compute the antiderivative $\int \frac{u-2}{u^2-u+1}\, du$ make the substitution $w = u^2 - u + 1$. Then $dw = (2u-1)\, du$ and the integral becomes

$$\int \frac{u-2}{u^2-u+1}\, du = \frac{1}{2}\int \frac{dw}{w} - \frac{3}{2}\int \frac{du}{u^2-u+1}.$$

The first integral on the right is

$$\int \frac{dw}{w} = \ln|w| + c_1,$$

and for the second we complete the square in the denominator to obtain

$$\int \frac{du}{u^2-u+1} = \int \frac{du}{\left(u-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{du}{\left(\frac{2}{\sqrt{3}}u - \frac{1}{\sqrt{3}}\right)^2+1} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}u - \frac{1}{\sqrt{3}}\right)+c_2.$$

So we have

$$\frac{2}{\sqrt{3}}\arctan\left(\frac{3}{\sqrt{3}}u - \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\ln|w| = \ln|x| + c,$$

where $u = \frac{y}{x}$ and $w = \left(\frac{y}{x}\right)^2 - \frac{y}{x} + 1$.

ii. $dy/dx = xy + xy^4$

$$\begin{aligned} \frac{dy}{dx} &= x(y+y^4) \\ \frac{dy}{y+y^4} &= xdx \\ \frac{dy}{y(1+y^3)} &= xdx \end{aligned}$$

Use partial fractions to express $\frac{1}{y(1+y^3)}$ in the form

$$\frac{1}{y(1+y^3)} = \frac{A}{y} + \frac{By^2+Cy+D}{1+y^3}$$

We find that $A = 1$, $B = -1$, $C = 0$, and $D = 0$.

$$\frac{1}{y(1+y^3)} = \frac{1}{y} - \frac{y^2}{1+y^3}$$

This means we can write

$$\begin{aligned}
\left(\frac{1}{y} - \frac{y^2}{1+y^3}\right) dy &= x dx \\
\ln|y| - \ln|1+y^3| &= \frac{1}{2}x^2 + c \\
\ln\left|\frac{y}{1+y^3}\right| &= \frac{1}{2}x^2 + c \\
\frac{y}{1+y^3} &= ae^{\frac{1}{2}x^2}.
\end{aligned}$$

Problem 7

i. $y'' - y' - 2y = t^2 e^{2t}$, $y(0) = 0$, $y'(0) = 1$

First find solutions to the homogeneous equation $y'' - y' - 2y = 0$. The associated characteristic polynomial is

$$r^2 - r - 2 = (r-2)(r+1).$$

Thus we have two independent complementary solutions $y_1 = e^{2t}$ and $y_2 = e^{-t}$. Now we write the particular solution as

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t),$$

for some functions u_1 and u_2. We determine u_1 and u_2 by solving the system

$$\begin{aligned}
u_1'y_1 + u_2'y_2 &= 0 \\
u_1'y_1' + u_2'y_2' &= t^2e^{2t}
\end{aligned}$$

This is

$$\begin{aligned}
u_1'e^{2t} + u_2'e^{-t} &= 0 \\
2u_1'e^{2t} - u_2'e^{-t} &= t^2e^{2t}
\end{aligned}$$

We find

$$u_1' = \frac{t^2}{3} \quad \text{and} \quad u_2' = -\frac{t^2}{3}e^{3t}.$$

Thus we may take

$$u_1 = t^3 \quad \text{and} \quad u_2 = -\left(\frac{1}{9}t^2 - \frac{2}{27}t + \frac{2}{81}\right)e^{3t}$$

Thus

$$y_p = t^3e^{2t} - \left(\frac{1}{9}t^2 - \frac{2}{27}t + \frac{2}{81}\right)e^{2t}.$$

We may drop the $\frac{2}{81}e^{2t}$ term since it appears in the homogeneous solution. The general solution $y = y_p + c_1 y_1 + c_2 y_2$ in this case may be written

$$y = \left(t^3 - \frac{1}{9}t^2 + \frac{2}{27}t\right)e^{2t} + c_1 e^{2t} + c_2 e^{-t}.$$

Initial conditions give $y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$. $y'(0) = 1 \Rightarrow \frac{2}{27} + 2c_1 - c_2 = 1 \Rightarrow c_1 = \frac{25}{81}$. So the final answer is

$$y = \left(t^3 - \frac{1}{9}t^2 + \frac{2}{27}t\right)e^{2t} + \frac{25}{81}e^{2t} - \frac{25}{81}e^{-t}.$$

ii. $y'' + y = -2\sin t$, $y(0) = 1$, $y'(0) = 1$

The characteristic equation is $r^2 + 1 = 0$. This gives homogeneous solutions $y_1 = \cos t$ and $y_2 = \sin t$. We solve the system

$$\begin{aligned} u_1' \cos t + u_2' \sin t &= 0 \\ -u_1' \sin t + u_2' \cos t &= -2\sin t \end{aligned}$$

which gives $u_1' = 2\sin^2 t$ and $u_2' = -2\sin t \cos t$. Using the identity $2\sin^2 t = 1 - \cos 2t$, we find

$$u_1 = t - \frac{1}{2}\sin 2t \quad \text{and} \quad u_2 = \cos^2 t.$$

The particular solution is thus

$$\begin{aligned} y_p &= \left(t - \frac{1}{2}\sin 2t\right)\cos t + \cos^2 t \sin t \\ &= (t - \sin t \cos t)\cos t + (1 - \sin^2 t)\sin t \\ &= t\cos t - \sin t \cos^2 t + \sin t - \sin^3 t \\ &= t\cos t - \sin t(1 - \sin^2 t) + \sin t - \sin^3 t \\ &= t\cos t \end{aligned}$$

so the general solution is

$$y = t\cos t + c_1 \cos t + c_2 \sin t.$$

The initial conditions show $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 1 \Rightarrow c_2 = 0$, so the final answer is

$$y = t\cos t + \cos t = (t+1)\cos t.$$

Problem 9

$y'' + 5y' + 6y = 0$

i. The characteristic equation is

$$r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0.$$

So the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-3t}.$$

ii. Put $x = y'$. Then $x' = y''$, so we obtain the first order system

$$\begin{cases} x' &= -5x - 6y \\ y' &= x \end{cases}$$

which may be expressed using matrices in the form $\vec{x}' = P\vec{x}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We find the eigenvalues by equating the determinant of $\lambda I - P$ to 0

$$\begin{vmatrix} -5-\lambda & -6 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda+5) + 6 = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2)$$

this has solutions $\lambda = -3$ and $\lambda = -2$. Substituting these into $(\lambda I - P)(\vec{v}) = 0$ we find solutions

$$\vec{v_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Problem 12

Without damping we have the equation $F = -ku$. Given that it takes 8lbs to stretch the spring $\frac{1}{2}$ft, we find that $-8 = -k\frac{1}{2}$, so $k = 16$ft·lb. We also know that the acceleration due to gravity is $32\text{ft}/s^2$. Using $F = ma$ where $F = 8$ is the weight of the object, we find its mass to be $m = 8/32 = 1/4$ slugs.

i. The motion of the mass is given by

$$\frac{1}{4}u'' + 2u' + 16u = \cos 3t,\ u(0) = \frac{1}{6}, u'(0) = 0$$

where $u(t)$ denotes the displacement from equilibrium (its natural dangling position) t seconds after it is released. We have $u'(0) = 0$ since it is released from rest.

ii. The characteristic equation is $\frac{1}{4}r^2 + 2r + 16 = 0$ which has roots $-4 \pm 4\sqrt{3}i$. Therefore the complimentary solution is $u_c = e^{-4t}\left[c_1 \cos(4\sqrt{3}t) + c_2 \sin(4\sqrt{3}t)\right]$.

Using the method of undetermined coefficients, we guess a sparticular solution to have the form

$$u_p = A\cos 3t + B\sin 3t.$$

Substituting this into the equation of motion gives

$$-\frac{9}{4}A\cos 3t - \frac{9}{4}B\sin 3t - 6A\sin 3t + 6B\cos 3t + 16A\cos 3t + 16B\sin 3t = \cos 3t.$$

This is equivalent to the system of equations

$$\begin{aligned} -\frac{9}{4}A + 6B + 16A &= 1 \\ -\frac{9}{4}B - 6A + 16B &= 0 \end{aligned}$$

we find $A = \frac{220}{3601}$ and $B = \frac{96}{3601}$. The general solution is given by

$$u = e^{-4t}\left[c_1\cos(4\sqrt{3}t) + c_2\sin(4\sqrt{3}t)\right] + A\cos 3t + B\sin 3t.$$

The initial conditions $u(0) = \frac{1}{6}$ and $u'(0) = 0$ yield the system

$$\begin{aligned} A + c_1 &= 1/6 \\ -4c_1 + 4\sqrt{3}c_2 + 3B &= 0 \end{aligned}$$

Solving gives $c_1 = \frac{2281}{31606}$ and $c_2 = \frac{1849}{64818}\sqrt{3}$.

Problem 13

The operational determinant is given by

$$L = \begin{vmatrix} D^2+1 & -D^2 \\ D^2-1 & D^2 \end{vmatrix} = (D^2+1)D^2 - -D^2(D^2-1) = 2D^4$$

This gives the equations

$$2D^4x = \begin{vmatrix} 2e^{-t} & -D^2 \\ 0 & D^2 \end{vmatrix} = 2e^{-t} \quad \text{and} \quad 2D^4y = \begin{vmatrix} D^2+1 & 2e^{-t} \\ D^2-1 & 0 \end{vmatrix} = 0$$

$D^4x = e^{-t}$ has the general solution $x = a_3t^3 + a_2t^2 + a_1t + a_0 + e^{-t}$ and $D^4y = 0$ has the general solution $y = b_3t^3 + b_2t^2 + b_1t + b_0$. Resubstituting these back into the second equation $(D^2-1)x + D^2y = 0$ yields

$$6a_3t + 2a_2 - a_3t^3 - a_2t^2 - a_1t - a_0 + 6b_3t + 2b_2 = 0.$$

Thus matching like terms gives

$$\begin{aligned} 2a_2 - a_0 + 2b_2 &= 0 \\ 6a_3 - a_1 + 6b_3 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \end{aligned}$$

We obtain the same system if we substitute into the first equation $(D^2+1)x - D^2 y = 2e^{-t}$. Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_0 + a_1 t + e^{-t} \\ b_0 + b_1 t + \frac{a_0}{2} t^2 + \frac{a_1}{6} t^3 \end{pmatrix}$$

where a_0, a_1, b_0, and b_1 are arbitrary constants.