Problem 1.

In a small town of 100 persons, there are $P(t)$ persons having the flu after $t$ days. Assume that the rate of increase of $P(t)$ satisfies the differential equation:

$$\frac{dP}{dt} = \frac{kP(100 - P)}{100},$$

where $k$ is some constant.

1. Should $k$ be positive or negative? Explain your answer.

2. Find the solution if at time $t = 0$ only one person has the flu.

3. When $t$ gets large, does $P(t)$ approach some fixed value? If yes, what is this value?
Problem 2. Find the solutions of the initial value problems:

• \[ \frac{dv}{dx} - \frac{1}{x}v = 2x, \quad v(1) = 0 \]

• \[ (1 + x) \frac{dy}{dx} = 4y, \quad y(0) = 1. \]

• \[(xy')^2 + y^2 = 1, \quad y(1) = 0 \text{ Hint: take the square root first} \]

• \[ \frac{dv}{dx} - \frac{1}{x}v = xv^6, \quad v(1) = 1 \]
Problem 3.

1. Check that the following differential equation is exact and find its general solution:

\[(4x - y)dx + (6y - x)dy = 0.\]

2. Show that the following equation is not exact:

\[(xy + y^2)dx + x^2dy = 0.\] (1)

3. Find the general solution of equation (1).
Problem 4. Suppose that the air resistance of a drop of water (of mass $m$) in free fall is proportional to the cube of its velocity, so that its equation of motion is

$$m \frac{dv}{dt} = -kv^3 - mg$$

where $h$ is the height of the drop of water, $k$ is some positive constant and $g = 9.8 m/s^2$ is the gravitational acceleration.

What is the terminal velocity of the drop of water (the velocity it ultimately reaches)?
Problem 5.

Consider the function

\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

and the associated differential equation

\[ \frac{dy}{dt} = \frac{xy}{x^2 + y^2} \]

1. Find the points where \( f(x, y) \) is discontinuous (remember to justify your answer). Compute \( \frac{\partial f}{\partial y} \), and the points where \( \frac{\partial f}{\partial y} \) is discontinuous.

2. For what values of \( a \) and \( b \) the Initial Value Problem (IVP) \( y(a) = b \) is guaranteed to have a unique solution? For what values \( a, b \) there is no guarantee that the IVP \( y(a) = b \) has a solution?

3. Plot the slope field for this differential equation.

4. Considering your answer for part 2 and the slope field you drew for 3, do you think any solutions with initial condition \( y(0) = 0 \) exist, and if so, do you think there is a unique solution?

5. Find the differential equation’s general solution (implicit form).
Problem 6.

Evaluate $dF(x, y)$ when $F(x, y)$ is the given 2-variable function. Then solve
the stated differential equation. If you know how to do these, there is no work
whatsoever

1. $F(x, y) = x^2 + xy + y^3$

2. $F(x, y) = x^2 \cos(y)$