Practice Midterm 2

MAT 131

Midterm 2 will cover material from sections 2.7-3.7 and 3.9

1. Calculate the derivatives of the following functions:

(a) $f(x) = 3x^3 + 4x^2 + 5x + \frac{6}{x}$ $= 3x^3 + 4x^2 + 5x + 6x^{-1}$

$$f'(x) = 3 \cdot 3x^2 + 4 \cdot 2x + 5 + 6(-1)x^{-2} = 9x^2 + 8x + 5 - \frac{6}{x^2}$$

(b) $f(x) = \sin^{10} x$ $\quad y = u^{10} \quad u = \sin x \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{du^{10}}{du}\frac{d\sin x}{dx} = 10u^9 \cos x =$

$$= 10(\sin x)^9 \cos x$$

(c) $f(x) = \frac{x^2+1}{x+5}$

$$f'(x) = \frac{(x^2+1)'(x+5) - (x^2+1)(x+5)'}{(x+5)^2} = \frac{2x(x+5) - (x^2+1)1}{(x+5)^2} = \frac{2x^2+10x-x^2-1}{(x+5)^2}$$

$$= \frac{x^2+10x-1}{(x+5)^2}$$

(d) $f(x) = 3^x \log_3(x+3)$ $\quad 3^x = e^{(\ln 3)x} \quad \log_3(x+3) = \frac{\ln(x+3)}{\ln 3}$

$$f'(x) = \left(e^{(\ln 3)x}\right)' \frac{\ln(x+3)}{\ln 3} + e^{(\ln 3)x} \frac{1}{\ln 3}\left(\ln(x+3)\right)' = e^{(\ln 3)x} \cdot \ln 3 \frac{\ln(x+3)}{\ln 3} +$$

$$+ e^{(\ln 3)x} \frac{1}{\ln 3}\frac{1}{(x+3)} = 3^x \ln(x+3) + 3^x \frac{1}{(\ln 3)(x+3)}$$

(e) $f(x) = \arctan(x^2+1)$

$y = \arctan u \quad u = x^2+1$

$$\frac{dy}{dx} = \frac{d\arctan u}{du}\frac{du}{dx} = \frac{1}{1+u^2}\,2x = \frac{1}{1+(x^2+1)^2} \cdot 2x$$

(f) $f(x) = x^{\sin x}$ $= y \quad \ln y = \ln x^{\sin x} = \sin x \ln x.$

$$\frac{y'}{y} = \frac{d\ln y}{dx} = \cos x \ln x + \sin x \cdot \frac{1}{x} \Rightarrow y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right) =$$

$$= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

(g) $f(x) = \frac{x^3(x+2)^{3/4}}{(x-1)^{3/2}(x+3)^{5/2}}$ $= y \quad \ln y = \ln \frac{x^3(x+2)^{\frac{3}{4}}}{(x-1)^{3/2}(x+3)^{\frac{5}{2}}} = 3\ln x + \frac{3}{4}\ln(x+2)$

$$- \frac{3}{2}\ln(x-1) - \frac{5}{2}\ln(x+3).$$

$$\frac{y'}{y} = \frac{3}{x} + \frac{3}{4}\frac{1}{x+2} - \frac{3}{4}\frac{1}{x-1} - \frac{5}{2}\frac{1}{x+3}$$

$$y' = \frac{x^3(x+2)^{\frac{3}{4}}}{(x-1)^{\frac{3}{2}}(x+3)^5}\left(\frac{3}{x} + \frac{3}{4}\frac{1}{x+2} - \frac{3}{4}\frac{1}{x-1} - \frac{5}{2}\frac{1}{x+3}\right)$$

2. (a) Find the linear function that best approximates $\tan x$ at the point $x = \frac{\pi}{4}$

$$(\tan x)' = \frac{1}{\cos^2 x}. \qquad \tan x' \Big|_{x=\frac{\pi}{4}} = \frac{1}{\left(\cos\frac{\pi}{4}\right)^2} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{\frac{2}{4}} = 2.$$

$$y = f(a) + f'(a)(x-a). \qquad a = \frac{\pi}{4}. \quad f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$y = 1 + 2\left(x - \frac{\pi}{4}\right) = 2x + 1 - \frac{\pi}{2}.$$

(b) Use (a) to find an approximate value of $\tan(\frac{\pi}{4} + 0.02)$.

$$\tan\left(\frac{\pi}{4} + 0.02\right) \sim f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(\cancel{\frac{\pi}{4}} + 0.02 - \cancel{\frac{\pi}{4}}\right) =$$

$$= 1 + 2 \cdot 0.02 = 1 + 0.04 = 1.04.$$

3. Find the tangent line to the curve at the point a:

(a) $\sqrt[3]{x} + \sqrt[3]{y} = 1, a = (8, -1)$

$$||$$

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} = 1 \qquad \frac{d}{dx}\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right) = \frac{d}{dx} 1 = 0$$

$$\cancel{\frac{1}{3}} x^{\frac{1}{3}-1} + \cancel{\frac{1}{3}} y^{\frac{1}{3}-1} \cdot y' = 0.$$

$$x^{-\frac{2}{3}} + y^{-\frac{2}{3}} y' = 0 \qquad y' = -\frac{x^{-\frac{2}{3}}}{y^{-\frac{2}{3}}}. \qquad x = 8 \quad y = -1$$

$$(-1)^{-\frac{2}{3}} = \frac{1}{(-1)^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{-1})^2} = \frac{1}{(-1)^2} = 1. \qquad y' = -\frac{8^{-\frac{2}{3}}}{(-1)^{-\frac{2}{3}}} = \boxed{-\frac{1}{4}.}$$

$$8^{-\frac{2}{3}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{(2)^2} = \frac{1}{4}$$

(b) $x^2 - xy + y^2 = 4, a = (2, 2)$

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx} 4 = 0.$$

$$\Downarrow$$

$$2x - 1 \cdot y - x \cdot y' + 2yy' = 0.$$

$$2x - y = xy' - 2yy' = (x - 2y)y'$$

$$y' = \frac{2x - y}{x - 2y} \qquad \begin{matrix} x = 2 \\ y = 2. \end{matrix} \qquad y' = \frac{2 \cdot 2 - 2}{2 - 2.2} = \frac{4-2}{2-4} = -1.$$

4. Suppose f and g are differentiable. Write the derivative of the function $F(x) = \frac{f(x)g(x)}{f(x)+g(x)}$ in terms of f, g, f', and g'.

$$F'(x) = \left(\frac{f(x)g(x)}{f(x)+g(x)}\right)' = \frac{(fg)'(f+g) - fg(f'+g')}{(f+g)^2} =$$

$$= \frac{(f'g+fg')(f+g) - fg(f'+g')}{(f+g)^2} = \frac{\cancel{ff'g} + f^2g' + f'g^2 + \cancel{fg'g} - \cancel{fgf'} - \cancel{fgg'}}{(f+g)^2} =$$

$$= \frac{f^2g' + f'g^2}{(f+g)^2}.$$

5. Let $f(x) = \frac{x^2+1}{x^2-1}$.

(a) Compute $f'(x), f''(x)$.

(b) For which values of x is f increasing? decreasing? concave up? down?

(c) Use the information above to sketch the graph of $f(x)$. Clearly mark maximums/minimums, inflection points, and asymptotes (if any). Do not forget to mark the units on the axes.

a). $f'(x) = \frac{2x(x^2-1)-(x^2+1)2x}{(x^2-1)^2} = \frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$

$f'' = \frac{-4(x^2-1)^2-(-4x)\,2(x^2-1)\cdot 2x}{(x^2-1)^4} = -4\,\frac{x^4-2x^2+1-4x^2(x^2-1)}{(x^2-1^4}\cdot =$

$= -4\,\frac{x^4-2x^2+1-4x^4+4x^2}{(x^2-1)^4} = -4\,\frac{-3x^4+2x^2+1}{(x^2-1)^4} =$

$= -4\,\frac{(1-x^2)(1+3x^2)}{(x^2-1)^4} = 4\,\frac{\cancel{(x^2-1)}(1+3x^2)}{(x^2-1)^{\cancel{4}3}} = 4\,\frac{1+3x^2}{(x^2-1)^3}.$

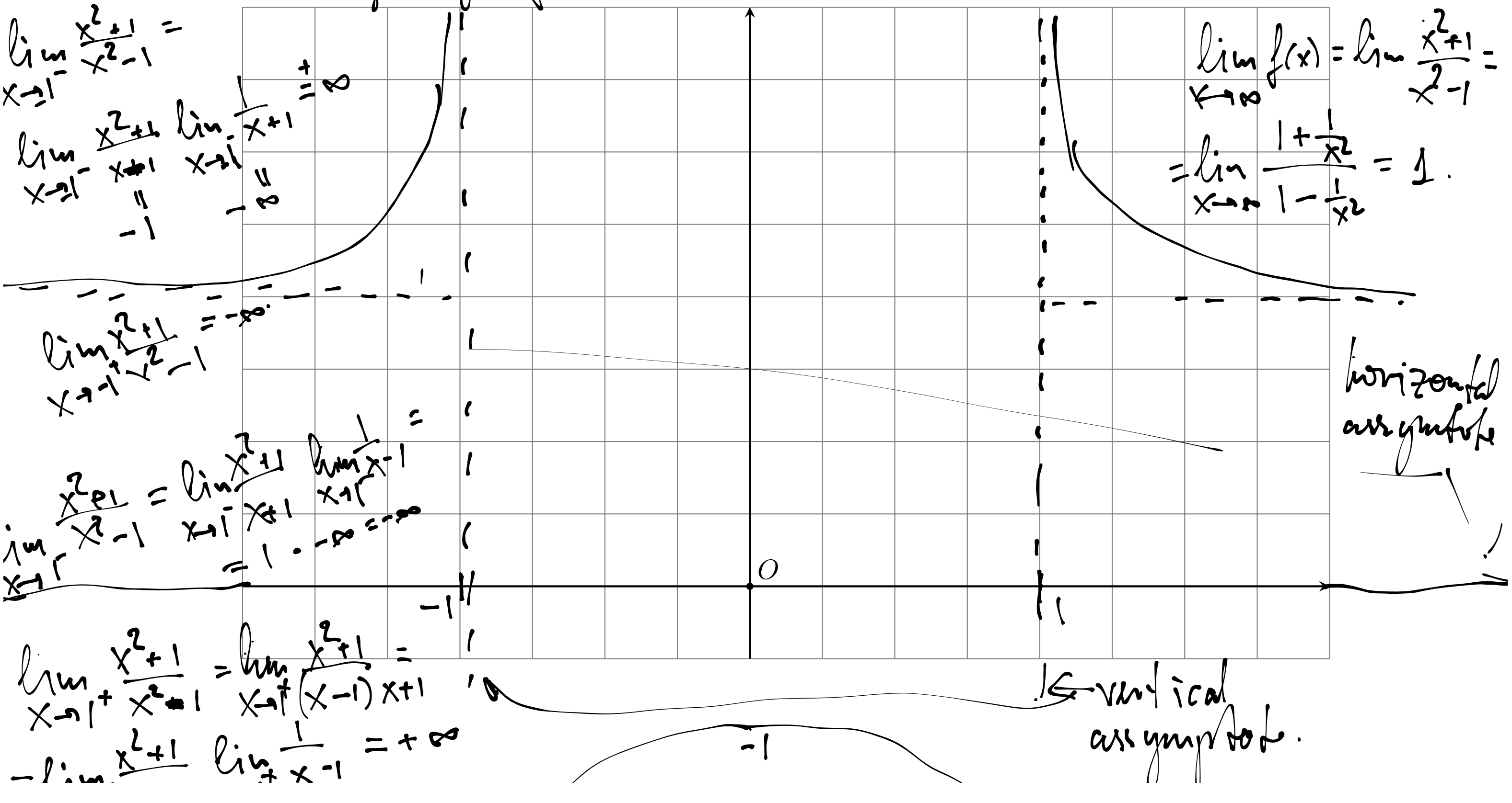
b) if $f'(x)>0 \Rightarrow f$ is increasing $\frac{-4x}{(x^2-1)^2}>0 \Leftrightarrow -4x>0 \Leftrightarrow x<0$

$f(x)$ is increasing if $x<0$ $f(x)$ decreasing $x>0$.

$f''(x)>0 \Rightarrow f$ is concave up. $4(1+3x^2)>0$. $4\,\frac{1+3x^2}{(x^2-1)^3}>0 \Leftrightarrow \frac{1}{(x^2-1)^3}>$

$\Leftrightarrow (x^2-1)^3>0 \Leftrightarrow x^2-1>0$ $x>1$ or $x<1$.

Analogousely $f''(x)<0$ f concave down $-1<x<1$



$\lim_{x\to-1^-}\frac{x^2+1}{x^2-1} = \lim_{x\to-1^-}\frac{x^2+1}{x-1}\,\lim_{x\to-1^-}\frac{1}{x+1} \overset{+}{=} \infty$

$\lim_{x\to-1^-}\frac{x^2+1}{x-1} = -1$, $\lim_{x\to-1^-}\frac{1}{x+1} = -\infty$

$\lim_{x\to-1^+}\frac{x^2+1}{x^2-1} = -\infty.$

$\lim_{x\to1^-}\frac{x^2+1}{x^2-1} = \lim_{x\to1^-}\frac{x^2+1}{x+1}\,\lim_{x\to1^-}\frac{1}{x-1} = = 1\cdot-\infty = -\infty$

$\lim_{x\to1^+}\frac{x^2+1}{x^2-1} = \lim_{x\to1^+}\frac{x^2+1}{(x-1)\,x+1} =$

$= \lim\frac{x^2+1}{\ }\,\lim_{+}\frac{1}{x-1} = +\infty$

$\lim_{x\to\infty} f(x) = \lim\frac{x^2+1}{x^2-1} =$

$= \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}} = 1.$

6. Let $f(x) = \sin x + \cos x$.

(a) Calculate $f'(x)$.

$f'(x) = \cos x - \sin x.$

$f''(x) = -\sin x - \cos x.$

$f'''(x) = -\cos x + \sin x.$

$f^{(4)}(x) = \sin x + \cos x.$

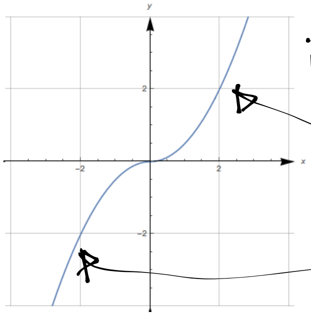
$f^{(8)}(x) = f^{(12)}(x) = \cdots = f^{(100)}(x) = \sin x + \cos x.$

(b) Calculate the 101st derivative of $f(x)$.

$f^{(101)}(x) = \cos x - \sin x.$

7. Match each graph of a function (first column) with the graph of its derivative (second column), by writing next to each graph of a function the corresponding letter.

b.

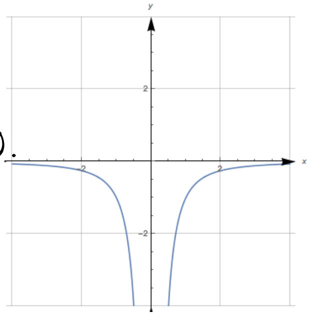


increasing

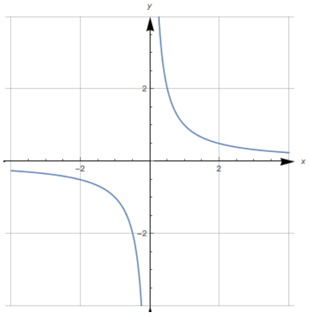
concave up. $f''>0$.

concave down. $f''<0$

a.

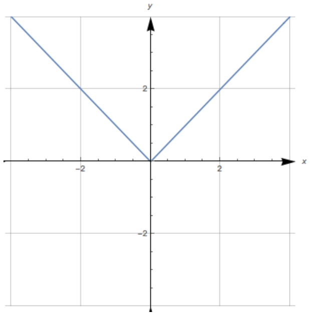


a

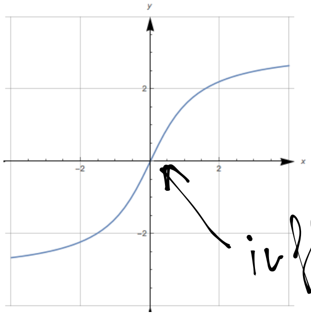


decreasing ⇒ $f'<0$

b.



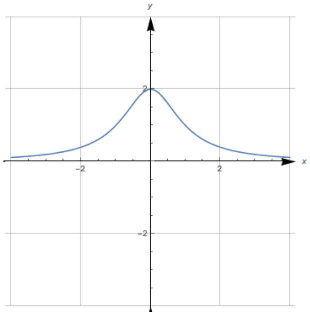
C



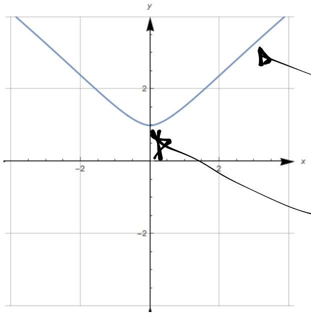
increasing.

inflection point.

c.



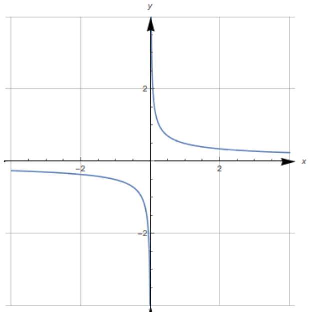
e



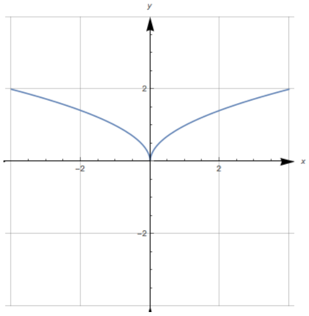
Concave up.

min.

d.



d.



e.