Practice Final Exam
MAT 131
December 2017

No notes, books or calculators.

You must show your reasoning, not just the answer.
Answers without justification will get only minimal partial credit.

Please cross out anything that is not part of your solution — e.g., some preliminary computations that you didn't need.

Answers should be simplified if possible (e.g., $\sin(0)$ **should be simplified to** 0**). However, do not round numerical answers unless the problem explicitly asks for it (e.g., do not replace** $1/3$ **by** 0.33**)**

1. (*20 pts*) Compute the following limits. If the limit does not exist, please specify whether the limit is equal to ∞, $-\infty$, or does not exist even allowing for infinite values.

(a) $\lim_{x\to 1^-} \frac{x^2+1}{x^2-1}$ $= \lim_{x\to 1^-} \frac{x^2+1}{(x-1)(x+1)} = \lim_{x\to 1^-} \frac{x^2+1}{x+1} \lim_{x\to 1^-} \frac{1}{x-1} = \frac{1+1}{1+1}(-\infty) = -\infty$

(b) $\lim_{x\to 0} \frac{\sin(x)}{2^x-1}$

$\sin(x) \to 0$, $2^x - 1 \to 0$ as $x \to 0$. Follows from continuity of functions. We can use l'Hospital's rule.

$= \lim_{x\to 0} \frac{\cos x}{2^x \cdot \ln 2} = \frac{1}{2^0 \ln 2} = \frac{1}{\ln 2}$.

(c) $\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3-h}}{h}$

$\sqrt{3+h} - \sqrt{3-h} \to 0$ as $h \to 0$. Can use l'Hospital's rule.

$= \lim_{h\to 0} \frac{(3+h)^{\frac{1}{2}} - (3-h)^{\frac{1}{2}}}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(3+h)^{-\frac{1}{2}} - \frac{1}{2}(3-h)^{-\frac{1}{2}}(-1)}{1} = \frac{1}{2} 3^{-\frac{1}{2}} + \frac{1}{2} 3^{-\frac{1}{2}} =$

$= \frac{1}{\sqrt{3}}$

(d) $\lim_{x\to\infty} \frac{x^3+2x+1}{3x^2+17x+219}$ $= \lim_{x\to\infty} \frac{\frac{x^3}{x^2} + \frac{2x}{x^2} + \frac{1}{x^2}}{\frac{3x^2}{x^2} + \frac{17x}{x^2} + \frac{219}{x^2}} = \lim_{x\to\infty} \frac{1}{3 + 17\frac{1}{x} + \frac{219}{x^2}}$ $\quad \lim x + \frac{2}{x} + \frac{1}{x^2}$

$= \frac{1}{3} \cdot \infty = \infty$

(e) $\lim_{x\to\infty} \frac{\sin(x^2)}{x^2+5}$ $= A$ $\quad -1 \le \sin y \le 1 \Rightarrow -1 \le \sin x^2 \le 1$

$-\frac{1}{x^2+5} \le \frac{\sin x^2}{x^2+5} \le \frac{1}{x^2+5} \Rightarrow \underbrace{-\lim_{x\to\infty} \frac{1}{x^2+5}}_{0} \le A \le \lim_{x\to\infty} \frac{1}{x^2+5} = 0 \Rightarrow A = 0$.

2. (*30 pts*) Compute the derivatives of the following functions.

(a) $f(x) = 2^x x^{3/2}$ $= e^{\ln 2x} \cdot x^{\frac{3}{2}}$

$f'(x) = e^{\ln 2x} \cdot \ln 2 \cdot x^{\frac{3}{2}} + e^{\ln 2x} \cdot \frac{3}{2} x^{\frac{3}{2}-1} = \ln 2 \cdot 2^x x^{\frac{3}{2}} + 2^x \cdot \frac{3}{2} x^{\frac{1}{2}}$.

(b) $f(x) = \tan(x^4 + \cos x)$ $= \tan(u) \quad u = x^4 + \cos x$

y //

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d \tan(u)}{du} \frac{d\, x^4 + \cos x}{dx} = \frac{1}{\cos^2 u} (4x^3 - \sin x) =$

$= \frac{4x^3 - \sin x}{\cos^2(x^4 + \cos x)}$

(c) $f(x) = \dfrac{x^2 + 7x}{(x+1)(x-3)}$ $= \frac{x(x+7)}{(x+1)(x-3)} \qquad \ln f = \ln x + \ln(x+7) - \ln(x+1) - \ln(x-3)$

$\frac{d}{dx} \ln f = \frac{f'}{f} = \frac{1}{x} + \frac{1}{x+7} - \frac{1}{x+1} - \frac{1}{x-3}$

$f' = \frac{x(x+7)}{(x+1)(x-3)} \left(\frac{1}{x} + \frac{1}{x+9} - \frac{1}{x+1} - \frac{1}{x-3} \right)$

(d) $f(x) = \arctan\sqrt{x^2-1}$ $= \arctan u$

y''

$u = \sqrt{v}$

$v = x^2 - 1$

$\frac{df}{dx} = \frac{d\arctan u}{du}\,\frac{du}{dv}\,\frac{dv}{dx} = \frac{1}{1+u^2}\,\frac{d\,v^{\frac{1}{2}}}{dv}\,\frac{d\,x^2-1}{dx} = \frac{1}{1+u^2}\cdot\frac{1}{2}v^{-\frac{1}{2}}$

$\cdot(2x) = \frac{1}{1+(\sqrt{x^2-1})^2}\,\frac{1}{\cancel{2}\sqrt{x^2-1}}\cdot\cancel{2}x = \frac{1}{1+|x^2-1|}\,\frac{x}{\sqrt{x^2-1}}$

(e) $g(u) = \dfrac{1}{u^2} + \sqrt{u} + u^3$ $= u^{-2} + u^{\frac{1}{2}} + u^3.$

$\frac{dg}{du} = -2u^{-3} + \frac{1}{2}u^{-\frac{1}{2}} + 3u^2 = -\frac{2}{u^3} + \frac{1}{2\sqrt{u}} + 3u^2$

(f) $f(t) = (\cos t)^{\tan t}$

$\ln f(t) = \tan t\,\ln(\cos t).$

$\frac{d\ln f}{dt} = \frac{f'}{f} = (\tan t)'(\ln\cos t) + \tan t(\ln\cos t)' = \frac{1}{\cos^2 t}\ln\cos t +$

$+ \tan t\,\frac{\sin t}{\cos t}.$

$f' = (\cos t)^{\tan t}\left(\frac{\ln\cos t}{\cos^2 t} + \tan^2 t\right)$

3. (*20 pts*) Let F be the function

$$F(x) = \frac{x^2}{x^3+4}.$$

(a) Find $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$.

$$\lim_{x\to\pm\infty} \frac{x^2}{x^3+4} = \lim_{x\to\pm\infty} \frac{\frac{x^2}{x^3}}{\frac{x^3+4}{x^3}} = \lim_{x\to\pm\infty} \frac{1}{x} \cdot \lim_{x\to\pm\infty} \frac{1}{1+\frac{4}{x^3}} = 0.$$

so $\lim_{x\to\pm\infty} \frac{x^2}{x^3+4} = 0$.

(b) The domain of F consists of all real numbers except one number a. What is a? Find $\lim_{x\to a^-} F(x)$ and $\lim_{x\to a^+} F(x)$.

$x^3+4 \neq 0$. nonvanishing denominator condition.

$a = (-4)^{\frac{1}{3}} = -\sqrt[3]{4}$

Sign of x^3+4: $-$ on the left of $-\sqrt[3]{x}$, $+$ on the right $\Rightarrow$ the sign of $\frac{x^2}{x^3+4}$: $-$ on the left of $-\sqrt[3]{x}$, $+$ on the right

we have a vertical assymptote $x = -\sqrt[3]{x}$.

$\lim_{x\to a^+} \frac{x^2}{x^3+4} = +\infty$ $\lim_{x\to a^-} \frac{x^2}{x^3+4} = -\infty$

(c) Find all the local extreme values of F and where they occur. For each of these values, specify whether it is a local maximum or a local minimum.

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2} = \frac{2x^4+8x-3x^4}{(x^3+4)^2} = \frac{8x-x^4}{(x^3+4)^2}.$$

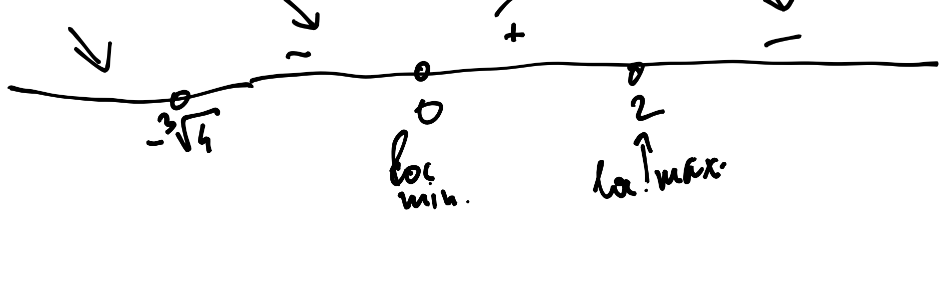
positive denominator.

Critical point. $x = -\sqrt[3]{4}$ and.

solution $8x - x^4 = 0$.

$x(8-x^3) = 0$ $x=0$
$x = 2$.

Sign of F'



(d) The second derivative of F is $F''(x) = \dfrac{2(x^6 - 28x^3 + 16)}{(4 + x^3)^3}$.

Show that the graph of F has an inflection point between $x = 0$ and $x = 1$.

Inflection point $F''(x) = 0$. $x^6 - 28x^3 + 16 = 0$ $z = x^3$

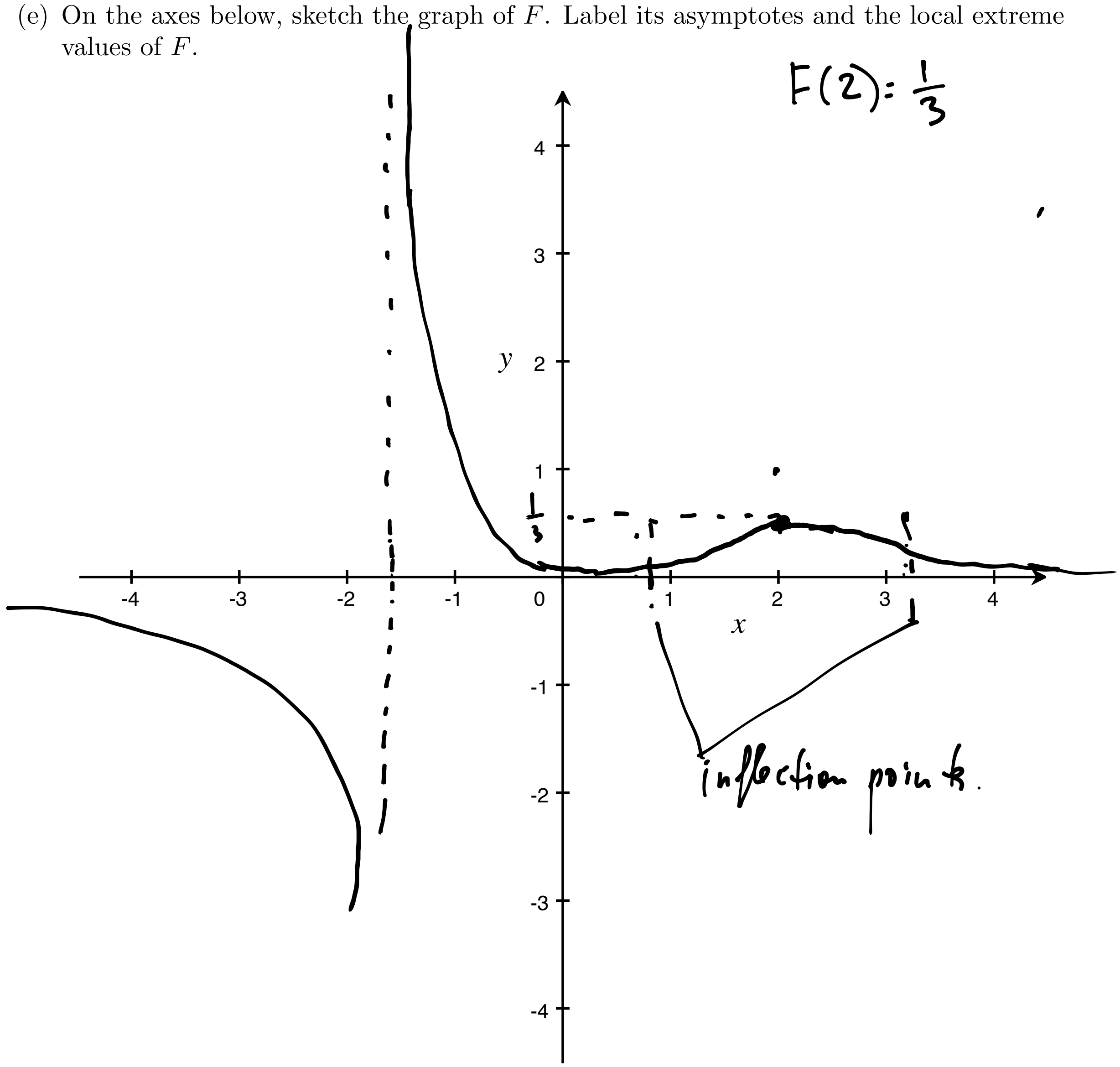
$z^2 - 28z + 16 = 0$ $z = \begin{cases} 2(7 - 3\sqrt{5}) \\ 2(7 + 3\sqrt{5}) \end{cases}$

Inflection points $x_1 = \sqrt[3]{2(7 - 3\sqrt{5})}$

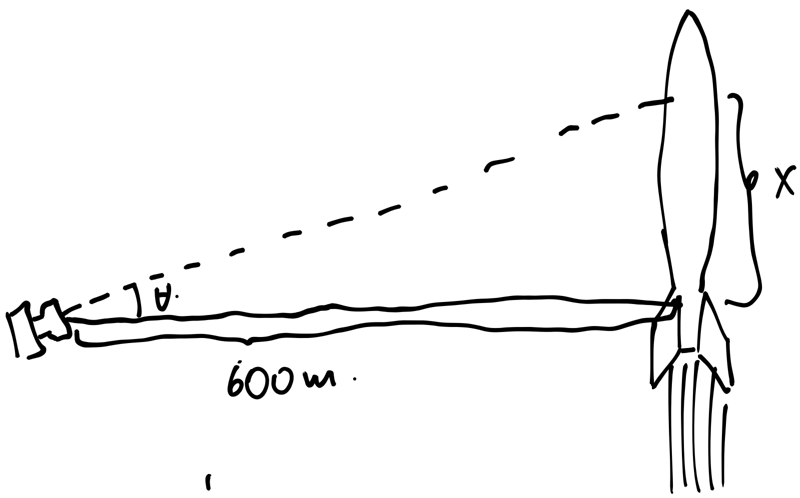
$x_2 = \sqrt[3]{2(7 + 3\sqrt{5})}$

Claim $0 < x_1 < 1 \iff 0 < \sqrt[3]{2(7 - 3\sqrt{5})} < 1 \iff 0 < 14 - 6\sqrt{5} < 1 \iff 6\sqrt{5} < 14$ and $13 < 6\sqrt{5} \iff 180 = 36 \cdot 5 < 196$ $169 < 180$.

(e) On the axes below, sketch the graph of F. Label its asymptotes and the local extreme values of F.



4. (*15 pts*) A TV camera is positioned 600 meters away from the rocket launch pad. The rocket is launched, rising vertically up, and the camera is kept aimed at the rocket, tracking its motion. At the moment the rocket's altitude is 600 meters, its speed is 100 m/s. How fast is the camera's angle of elevation changing at this moment? (The answer should be written in rad/s).



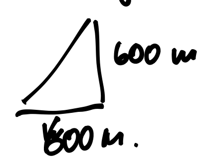
$\tan\theta = \frac{x}{600}$.

$(\tan\theta)' = \left(\frac{x}{600}\right)' = \frac{x'}{600}$

$\frac{1}{\cos^2\theta}\theta'$

$\theta' = \cos^2\theta \, \frac{x'}{600} =$

At the time $x = 600$. when the angle $\theta = \frac{\pi}{4}$



$\cos^2\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

$\theta' = \frac{1}{2} \cdot \frac{100}{600} = \frac{1}{12}$ rad/sec.

5. (*10 pts*) Find a linearization of the function $f(x) = e^{-x}\sin(\pi x)$ near the point $x = 2$.

$y = f(x_0) + f'(x_0)(x - x_0)$. $x_0 = 2$.

$f(x_0) = e^{-2}\underbrace{\sin(\pi \cdot 2)}_{=0} = 0$.

$f'(x) = -e^{-x}\sin(\pi x) + e^{-x}(\cos(\pi x)\cdot\pi)$.

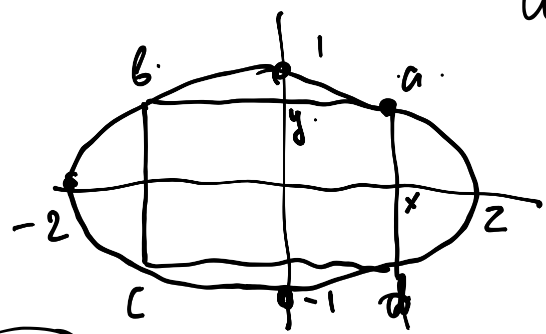
$f'(2) = -e^{-2}\underbrace{\sin(2\pi)}_{=0} + e^{-2}\cos(2\pi)\pi = \pi e^{-2}$

$y = \pi e^{-2}(x-2)$

6. (*15 pts*) Maximize the area of a rectangle that can be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

[Hint: instead of maximizing the area A, you can maximize A^2]



Coordinates of a are (x, y). C $(-x, -y)$
b are $(-x, y)$ d $(x, -y)$

$|ba| = 2x$ The area is $4xy$.
$|ad| = 2y$.

(x, y) satisfy $\frac{x^2}{4} + y^2 = 1$.

$B = A^2 = 16x^2y^2$ is maximal $\iff$ $A = 4xy$ is maximal.

Denote $x^2 = X$, $y^2 = Y$ then $\frac{X}{4} + Y = 1$, $X \geq 0$, $Y \geq 0$ and $A^2 = 16XY =$

$X = 4(1-Y)$, $Y \geq 0 \Downarrow Y \leq 1$

$= 16 \cdot 4(1-Y)Y =$
$= 16 \cdot 4(Y - Y^2)$.

$\frac{dB}{dY} = 16 \cdot 4(1-2Y) = 0$ $Y = \frac{1}{2}$.

$X = 4(1 - \frac{1}{2}) = 2$.

$+ \nearrow$ $\frac{1}{2}$ $- \searrow$

$\frac{1}{2}$ local max

$B\Big|_{Y=\frac{1}{2}} = 16 \cdot 4 \cdot \left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) = 16 \cdot 4 \cdot \frac{1}{4} = 16$ ← global maximum.

$B\Big|_{Y=0} = 0$ $B\Big|_{Y=1} = 0$.

$y = \sqrt{Y} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

$x = \sqrt{X} = \sqrt{2} = \sqrt{2}$.

$A = \sqrt{B} = \sqrt{16} = 4$ – the value of the maximal area.

7. (*20 pts*) Compute each of the following (definite or indefinite) integrals.

(a) $\int_0^2 (x^3+2x+1)\,dx$ $= \frac{x^4}{4}+x^2+x\Big|_0^2 = \frac{2^4}{4}+2^2+2.$

(b) $\int (3e^x+2\sin x)\,dx$ $= 3e^x + 2(-\cos x) + C$

(c) $\int \sqrt{3x+1}\,dx$ $= \int u^{\frac12}\,\frac{du}{3} = \frac{u^{\frac12+1}}{\left(\frac12+1\right)3} + C = \frac{2u^{\frac32}}{3\cdot 3} + C.$

$3x+1=u$

$3\,dx = du$

$dx = \frac{du}{3}$

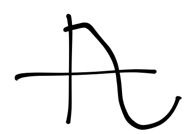
(d) $\int_{-\pi}^{\pi} x\cos(\frac{1}{x})dx$

$(-x)\cos\left(-\frac{1}{x}\right) = -x\cos\frac{1}{x}$ odd function.

even function.

Symmetric region.

$\int_{-\pi}^{\pi} x\cos\frac{1}{x}\,dx = 0.$



(e) $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$ $= \int_0^{\pi} \sin u\,\frac{du}{2} = -\frac{\cos u}{2}\Big|_0^{\pi} = -\frac12(-1-1) = 1.$

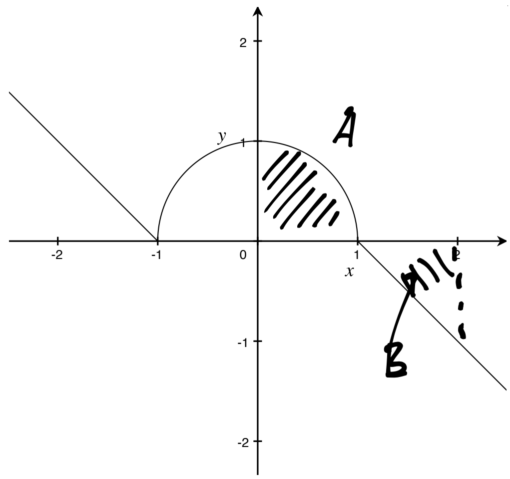
$x^2=u$

$2x\,dx = du.$

$u(0)=0$

$u(\sqrt{\pi}) = (\sqrt{\pi})^2 = \pi.$

8. (*10 pts*) Below is the graph of a continuous function f. The graph is composed of the upper half of the unit circle and two half-lines with slope -1.



(a) Compute $\int_0^2 f(x)\,dx$. (*Hint:* You do not need to find an explicit antiderivative for f.)

$\int_0^2 f(x)\,dx = A - B = \frac{\pi \cdot 1}{4} - \frac{1\cdot 1}{2} = \frac{\pi}{4} - \frac{1}{2}$

(b) Define $g(x) = \int_{-3}^{x^2} f(t)\,dt$. What are the critical points of g? For each critical point c, determine whether $g(c)$ is a local minimum, a local maximum, or neither.
(*Note:* Critical points are also called "critical numbers".)

$g(x)' = f(x^2)\cdot 2x$ critical points of g are solutions

$f(x^2)2x = 0$. $x=0$. $f(z)=0 \Rightarrow z=1$. $z>0$. $x^2=1$ $x=\pm1$.

$+$ loc max $-$ loc min $+$ loc max $-$

$\nearrow$ -1 $\searrow$ 0 $\nearrow$ 1 $\searrow$

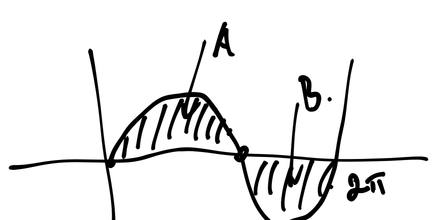
Sign of $2x f(x^2)$.

9. (*10 pts*) Write the first iteration of the Newton's method for the problem of finding a solution of $\log(x) + 10 = x^2$.

$f(x) = x^2 - 10 - \log x$ $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} =$

$= x_n - \frac{x_n^2 - 10 - \ln x_n}{2x_n - \frac{1}{x_n}}$ $\quad x_1 = 1$ $\quad x_2 = 1 - \frac{1^2 - 10 - \ln 1}{2\cdot 1 - 1} = 1 - (1-10) = 10.$

10. (*10 pts*) Find the area enclosed by the x-axis, the graph of $y = \sin(x)$, y-axis and $x = 2\pi$.



The area $= A + B =$

$= \int_0^\pi \sin x\,dx + \int_\pi^{2\pi} |\sin x|\,dx = \int \sin x - \int_\pi^{2\pi} \sin x$

negative well

pos

$= -\cos x\Big|_0^\pi - (-\cos x)\Big|_\pi^{2\pi} = -(-1-1) + \cos x\Big|_\pi^{2\pi} = 2 + (1-(-1)) = 2+2 = 4.$

give +1