Practice Midterm 1 Solutions

MAT 125, Spring 2017

Name:	ID #:	Section:

Please answer each question in the space provided. Show your work whenever possible. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer.

(1) Calculate the following limits

(a) $\lim_{x\to 2} 3x^2 + x - 2$

Solution: Since $f(x) = 3x^2 + x - 2$ is continuous, $\lim_{x\to 2} f(x) = f(2) = 12$.

(b) $\lim_{y\to -3} |y+3|$

Solution: For $y > -3$, $y + 3 > 0$ so $|y+3| = y+3$. Thus, $\lim_{y\to(-3)+} |y+3| = \lim_{y\to(-3)+} y+3 = (-3)+3 = 0$. Similarly, $\lim_{y\to(-3)-} |y+3| = \lim_{y\to(-3)-} -(y+3) = -((-3)+3) = 0$. Since one-sided limits coincide, $\lim_{y\to -3} |y+3| = 0$.

(c) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

Solution: In this case, plugging in $x = 2$ is impossible because in this case both the numerator and denominator are zero. Instead, we can factor the numerator, using the formula for roots of quadratic equation, to get

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\frac{(x+3)(x-2)}{x-2}$$
$$= \lim_{x\to 2}(x+3) = 5$$

(d) $\lim_{q\to 2} \frac{2q^2+5}{\sqrt{q+2}}$

Solution: This function is continuous, thus $\lim_{q\to 2} f(q) = f(2) = (2\cdot 4+5)/\sqrt{4} = 13/2 = 6.5$

(e) $\lim_{t\to 3} \frac{\sqrt{t}-\sqrt{3}}{t-3}$

Solution: Again, both numerator and denominator have limit zero, so we can not use the quotient rule; instead, we can multiply both numerator and denominator by $\sqrt{t}+\sqrt{3}$:

$$\lim_{t\to 3}\frac{\sqrt{t}-\sqrt{3}}{t-3} = \lim_{t\to 3}\frac{(\sqrt{t}-\sqrt{3})(\sqrt{t}+\sqrt{3})}{(t-3)(\sqrt{t}+\sqrt{3})}$$
$$= \lim_{t\to 3}\frac{t-3}{(t-3)(\sqrt{t}+\sqrt{3})} = \lim_{t\to 3}\frac{1}{\sqrt{t}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

(f) $\lim_{s\to 0} s^2\cos\left(s+\frac{1}{s}\right)$

Solution: Denote $f(s) = s^2\cos\left(s+\frac{1}{s}\right)$.
Since $-1 \le \cos\left(s+\frac{1}{s}\right) \le 1$, we have $-s^2 \le f(s) \le s^2$. Since $\lim_{s\to 0} s^2 = \lim_{s\to 0}(-s^2) = 0$, by squeeze theorem we have $\lim_{s\to 0} f(s) = 0$.

(2) Calculate

$$\lim_{x\to(\pi/2)-} \frac{1+\tan x}{1-\tan x}$$

Solution: First, let us see what happens with $t = \tan x$ as $x \to (\pi/2)-$. By definition, $\tan x = \sin x/\cos x$. As $x \to (\pi/2)-$, we know that $\sin x \to 1$ and $\cos x \to 0$. Thus, we can't use quotient rule to compute the limit of $\tan x$ (in fact, this limit does not exist).

However, we can rewrite the expression as follows:

$$\lim_{x\to(\pi/2)-} \frac{1+\tan x}{1-\tan x} = \lim_{x\to(\pi/2)-} \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}$$
$$= \lim_{x\to(\pi/2)-} \frac{\cos x + \sin x}{\cos x - \sin x}$$

(by multiplying both numerator and denominator by $\cos x$). Now we can use the quotient rule: since $\sin, \cos$ are continuous, as $x \to (\pi/2)-$, we have

$$\sin x \to \sin(\pi/2) = 1$$
$$\cos x \to \cos(\pi/2) = 0$$

so

$$\lim_{x\to(\pi/2)-} \frac{\cos x + \sin x}{\cos x - \sin x} = \frac{0+1}{0-1} = -1$$

Note: this is a difficult problem — I probably wouldn't include such a problem in the actual exam.

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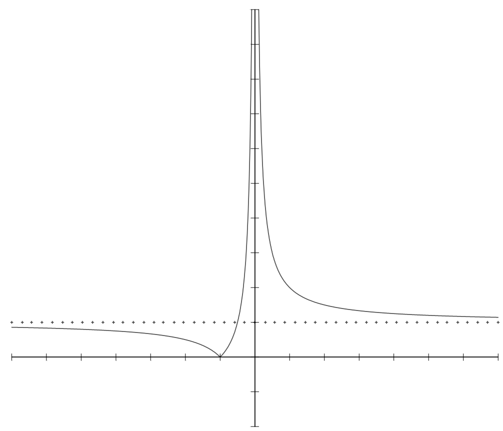
(3) Let $f(x) = \left|1 + \frac{1}{x}\right|$.
(a) Sketch the graph of f and identify the asymptotes.
(b) Find all values of x for which f is not continuous.

Solution: The graph is shown below; it is obtained from the graph of $y = \frac{1}{x}$ by shifting it one unit up (this gives graph of $y = 1 + \frac{1}{x}$) and then reflecting the part of the graph below the x-axis.



The asymptotes are: horizontal: $y = 1$ and vertical: $x = 0$.

Since the functions $1/x$ and $|x|$ are continuous, $f(x)$ is also continuous. Thus, the only discontinuity points are when the function is not defined, that is, at $x = 0$.

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(4) Find

$$\lim_{x \to 1} e^{(x^2 - x - 1)}.$$

Between which two integers (whole numbers) does the answer lie?

Solution: Since this function is continuous, $\lim_{x \to 1} e^{(x^2-x-1)} = e^{1^2-1-1} = e^{-1} = 1/e$. Since $e \approx 2.7\ldots$, $0 < 1/e < 1$.

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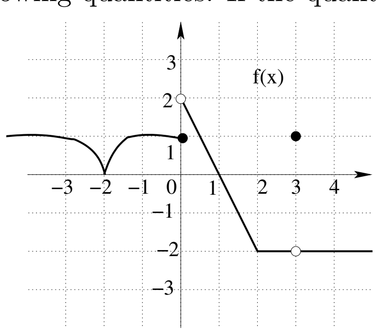
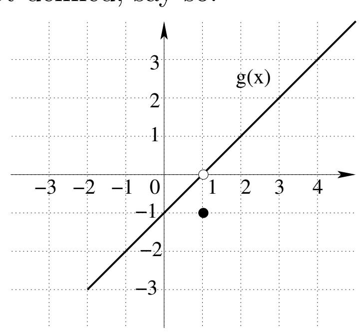
(5) Use the graphs of $f(x)$ and $g(x)$ below to compute each of the following quantities. If the quantity is not defined, say so.





$f(0)$	$\lim_{x\to 0+} f(x)$	$\lim_{x\to 0-} f(x)$	$\lim_{x\to 0} f(x)$
$\lim_{x\to 1} g(x)$	$\lim_{x\to 1} f(x) - g(x)$	$\lim_{x\to 3}(2f(x) - f(3))$	

Solution: $f(0) = 1$; $\lim_{x\to 0+} f(x) = 2$; $\lim_{x\to 0-} f(x) = 1$;
$\lim_{x\to 0} f(x)$ does not exist, since the one-sided limits are different;
$\lim_{x\to 1} g(x) = 0$;
$\lim_{x\to 1} f(x) - g(x) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 0 - 0 = 0$;
$\lim_{x\to 3}(2f(x) - f(3)) = \left(2\lim_{x\to 3} f(x)\right) - f(3) = 2(-2) - 1 = -5.$

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(6) Consider the function

$$f(t) = \begin{cases} \frac{t}{t-1} & t \geq 0 \\ t+1 & t < 0 \end{cases}$$

(a) At which points is this function continuous?
(b) Find the left and right limits, if they exist, at $t = 0$.

Solution:
For $t < 0$, this function is given by $f(t) = t + 1$, so it is continuos.
For $t > 0$, this function is given by $f(t) = \frac{t}{t-1}$, so it is continuos wherever defined. Thus, it is continuous at all points where denominator is non-zero, i.e. $t \neq 1$
It remians to consider the point $t = 0$. At this point, function is defined by different formulas on two sides of this point. To check whether it is continuous, we compute the one-sided limits.

$$\lim_{t\to 0+} f(t) = \lim_{t\to 0+} \frac{t}{t-1} = \frac{0}{0-1} = 0$$
$$\lim_{t\to 0-} f(t) = \lim_{t\to 0-} (t+1) = 0 + 1 = 1$$

Since these limtis are not equal, limit $\lim_{t\to 0} f(t)$ does not exist. So $f(t)$ is not continuous at 0.
Thus, $f(t)$ is continuous everywhere except $t = 0$, $t = 1$.

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(7) Find an interval of length 1 which contains the root of the following function. Please remember to write the justification **why** this interval contains the root, not just the answer!
$f(x) = x^3 - \frac{1}{x+1}$

Solution: $f(x)$ is a rational function. It is continuous for all x that are in the domain. In our case these are all $x \neq -1$. The values of $f(x)$ for $x = 0$ and $x = 1$ are -1 and $1/2$ respectively. The function $f(x)$ is continuous on $[0, 1]$ and $f(0) < 0, f(1) > 0$. By intermediate value theorem there is $c \in [0, 1]$ such that $f(c) = 0$.

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(8) Suppose $f(x) = 3 + \frac{1}{2x+1}$
(a) Compute $f'(1)$ using the definition of derivative (without using the power rule or other rules for computing derivatives - even if you

know them!)
(b) Write the equation of the tangent line to the graph of this function at $x = 1$.

Solution: (a)

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(\frac{1}{2(h+1)+1} + 3) - (\frac{1}{2(1)+1} + 3)}{h} =$$

$$\lim_{h\to 0} \frac{(\frac{1}{2(h+1)+1} + 3) - 10/3}{h} = \lim_{h\to 0} -\frac{2h}{h3(2h+3)} = -\lim_{h\to 0} \frac{2}{3(2h+3)} = -2/9$$

(b) The tangent line to the graph through $x = 1$ has equation $y = f(1) + f'(1)(x-1)$. As $f(1) = 10/3$, equation simplifies $y = 10/3 - 2/9(x-1)$.

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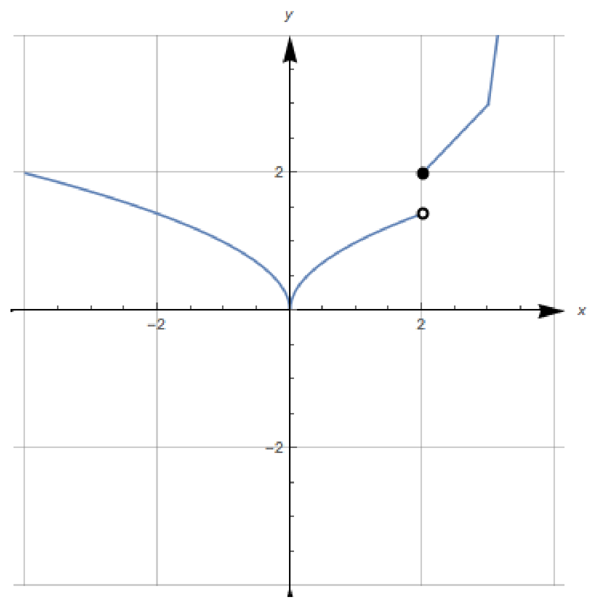
(9) Determine the points where the function $y = f(x)$ whose graph is given below is not differentiable



Solution: The problematic points are $x = 0, 2, 3$. At all other points the graph is smooth.Let us understand what is wrong with $0, 2, 3$. At the point $x = 2$ the function is discontinuous. We know that if a function differentiable at a point it is automatically continuous at this point. Thus f is not differentiable at $x = 2$. At the point 0 the tangent line is vertical, therefore its slope is infinite. The derivative doesn't have a finite value. At the point $x = 3$, the limits $\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h}$ and $\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h}$ do not coincide. So $f'(3)$ is undefined.