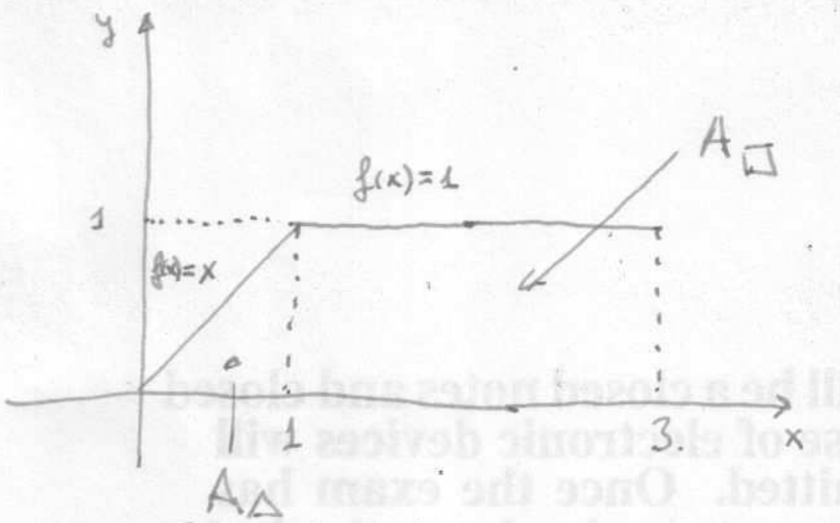
Problem 1 Evaluate the integral by interpreting it as an area:

$$\int_0^3 f(x)dx,$$

where

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 \le x \le 3. \end{cases}$$

The graph of function is



The integral is the area A under the graph

$$A = A_\triangle + A_\square.$$

$$A_\triangle = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}.$$

base of triangle

height of △.

$$A_\square = 1 \cdot (3-1) = 2.$$

$$\int_0^3 f(x)dx = \frac{1}{2} + 2 = 2\frac{1}{2}.$$

Problem 2 Evaluate

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(4+\frac{5i}{n}\right)^{1/2}\frac{5}{n} \quad (\times)$$

by interpreting the limit as an integral and using the Evaluation Theorem to compute this integral.

Let $f: [a\,b] \to \mathbb{R}$ be a cont. function.

By definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}i\right) \quad (\times\times)$$

We compare two formulas $(\times)$ and $(\times\times)$ in our case the function $f(x) = x^{\frac{1}{2}}$.

$$4+\frac{5i}{n} = a+\frac{b-a}{n}i$$

We conclude that $4 = a \quad b-a=5$

$$\Rightarrow b = 5+a = 5+4 = 9.$$

The limit is equal to

$$\int_a^b f(x)\,dx = \int_4^9 \sqrt{x}\,dx$$

Problem 3 Evaluate the following definite integral:

$$\int_2^4 \frac{x^3\sqrt{x^5} - x^2\sqrt[3]{x^2})}{x^4}dx$$

We simplify first

$$\frac{x^3\sqrt{x^5} - x^2\sqrt[3]{x^2}}{x^4} = x^{-4}\left(x^3 x^{\frac{5}{2}} - x^2 x^{\frac{2}{3}}\right) =$$

$$= x^{3+\frac{5}{2}-4} - x^{2+\frac{2}{3}-4} = x^{\frac{6+5-8}{2}} - x^{\frac{6+2-12}{3}} =$$

$$= x^{\frac{3}{2}} - x^{-\frac{4}{3}}$$

Antiderivative.

$$\int_2^4 \left(x^{\frac{3}{2}} - x^{-\frac{4}{3}}\right)dx = \left(\frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} - \frac{1}{-\frac{4}{3}+1} x^{-\frac{4}{3}+1}\right)\Bigg|_2^4$$

$$= \left(\frac{2}{5} x^{\frac{5}{2}} + 3x^{-\frac{1}{3}}\right)\Bigg|_2^4 =$$

$$= \frac{2}{5}\, 4^{\frac{5}{2}} + 3(4)^{-\frac{1}{3}} - \left(\frac{2}{5}\, 2^{\frac{5}{2}} + 3\, 2^{-\frac{1}{3}}\right).$$

Problem. *Let f(x) and g(x) be two functions such that:*
$\int_{-1}^{2}[f(x)+g(x)]dx = 3, \int_{-1}^{2}[f(x)-2g(x)]dx = 1, \int_{-1}^{0} f(x)dx = -1$
Find $\int_0^2 f(x)dx$

Proof.

$$\int_0^2 f(x)dx = \int_{-1}^{2} f(x)dx - \int_{-1}^{0} f(x)dx \tag{1}$$
$$= \int_{-1}^{2} f(x)dx - (-1) \tag{2}$$

and we have:

$$\int_{-1}^{2} f(x)dx = \frac{1}{3}\left(2\int_{-1}^{2}[f(x)+g(x)]dx + \int_{-1}^{2}[f(x)-2g(x)]dx\right) \tag{3}$$
$$= \frac{1}{3}(2\cdot 3+1) \tag{4}$$
$$= \frac{7}{3} \tag{5}$$

so we can conclude that:
$\int_0^2 f(x)dx = \frac{7}{3} + 1 = \frac{10}{3}$

Problem. *1) find* $\frac{d}{dx}\left(e^{x^2}\right)$
2)Evaluate
$\int_0^2 xe^{x^2}dx$

Proof. 1) we use the chain rule to get: $\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$

2) Note that by part (1), the antiderivative of xe^{x^2} is $\frac{1}{2}e^{x^2}$. Now apply the FUNDAMENTAL THEOREM OF CALCULUS to get:
$\int_0^2 xe^{x^2}dx = \frac{1}{2}e^{x^2}\Big]_0^2 = \frac{1}{2}\left(e^4 - e^0\right) = \frac{1}{2}\left(e^4 - 1\right)$

Problem. *Find the antiderivative of:*
1. $\frac{sin(2x)}{cos(x)}$
2. $e^{x+7}2^{-2x}$
3. $\frac{x^2}{x^3}$

Proof. 1. Use the 'double angle formula': $sin(2x) = 2sin(x)cos(x)$ and then the problem becomes VERY simple:
$\int \frac{sin(2x)}{cos(x)}dx = \int \frac{2sin(x)cos(x)}{cos(x)}dx = \int 2sin(x)dx = -2cos(x) + C$
2. So this problem seems like it may contain integration by part or substitution, but if we SIMPLIFY well we can avoid both:
$e^{x+7}2^{-2x} = e^7e^x\left(\frac{1}{4}\right)^x = e^7\left(\frac{e}{4}\right)^x$

Now there is only ONE function and it is NOT composite thus:

$$\begin{aligned}\int e^{x+7}2^{-2x}dx &= \int e^7 \left(\frac{e}{4}\right)^x dx && (6)\\ &= e^7 \int \left(\frac{e}{4}\right)^x dx && (7)\\ &= e^7 \frac{1}{\ln\left(\frac{e}{4}\right)} \left(\frac{e}{4}\right)^x + C && (8)\\ &= \frac{e^7}{1-\ln(4)} \left(\frac{e}{4}\right)^x + C && (9)\end{aligned}$$

3. If there is one thing you should be noticing by now, it is this: TRY TO REDUCE THE FUNCTION BEFORE INTEGRATING OR DERIVATING! This last one is very simple one line calculation if we just notice that:

$\frac{x^2}{x^3} = x^{-1}$ (REDUCTION OF FRACTIONS HAS COME UP A LOT SO PAY ATTENTION TO IT!)

And so

$\int \frac{x^2}{x^3} dx = \int x^{-1} dx = \ln(x) + C$

Problem 7 Compute the derivative of the function

$$\ln(\tan^2(x))$$

and simplify your answer.

$$\left(\ln(\tan^2(x)) \right)'$$

$$= \frac{1}{\tan^2(x)} \cdot (\tan^2 x)'$$

$$= \frac{2\tan(x)}{\tan^2(x)} \cdot (\tan x)'$$

$$= \frac{2}{\tan(x)} \sec^2 x$$

$$= \frac{2}{\frac{\sin x}{\cos x} \cdot \cos^2 x} = \frac{2}{\sin x \cos x} \quad \left(= \frac{4}{\sin 2x} \right)$$

$$\left(= 4 \csc 2x \right)$$

Problem 8 Estimate the integral

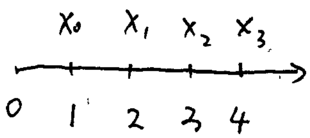
$$\int_1^4 x \ln x dx$$

using three rectangles and

a) right endpoints b) left endpoints.

c) Are your answers in a) and b) over- or under-estimates of the actual integral?

(Hint: you may want to determine whether the function $f(x) = x \ln x$ is increasing or decreasing)



$\Delta x = 1$

a) $R_3 = \sum_{i=1}^{3} x_i \ln x_i = 2\ln 2 + 3\ln 3 + 4\ln 4$

$= 10\ln 2 + 3\ln 3$

b) $L_3 = \sum_{i=0}^{2} x_i \ln x_i = 1\ln 1 + 2\ln 2 + 3\ln 3$

$= 2\ln 2 + 3\ln 3$

c) Since $(x\ln x)' = \ln x + 1 > 0$ on $[1,4]$,

it's increasing.

So. R_3 is overestimate &

L_3 is underestimate.