H.W. #7 Solution Key:

5.6 #4,8,10,14,18,22,26,28,34,36,40,44

$\int u\,dv = uv - \int v\,du$

4-) $\int xe^{-x}dx \rightarrow \begin{array}{ll} u=x & dv=e^{-x}dx \\ du=dx & v=-e^{-x} \end{array} \rightarrow -xe^{-x} + \int e^{-x}dx = -xe^{-x} + (-e^{-x}+C) = -e^{-x}(x+1)+C$

8-) $\int x^2 \cos mx\,dx \rightarrow \begin{array}{ll} u=x^2 & dv=\cos mx\,dx \\ du=2x\,dx & v=\frac{\sin mx}{m} \end{array} \rightarrow \frac{x^2\sin mx}{m} - \frac{2}{m}\int x\sin mx\,dx$

$\int x\sin mx\,dx \rightarrow \begin{array}{ll} u=x & dv=\sin mx\,dx \\ du=dx & v=\frac{\sin mx}{m} \end{array} \rightarrow \frac{x^2\sin mx}{m} - \frac{2x\sin mx}{m^2} + \frac{2}{m^2}\int \sin mx\,dx$

Apply integration by parts twice

$\frac{x^2\sin mx}{m} - \frac{2x\sin mx}{m^2} + \frac{2\cos mx}{m^3} + C$

10-) $\int p^5 \ln p\,dp \rightarrow \begin{array}{ll} u=\ln p & dv=p^5dp \\ du=\frac{1}{p}dp & v=\frac{p^6}{6} \end{array} \rightarrow \frac{p^6}{6}\ln p - \int \frac{p^5}{6}dp = \frac{p^6}{6}\ln p - \frac{p^6}{36} + C$

14-) (i) $\boxed{\int e^{-\theta}\cos 2\theta} \rightarrow \begin{array}{ll} u=e^{-\theta} & dv=\cos 2\theta\,d\theta \\ du=-e^{-\theta}d\theta & v=\frac{\sin 2\theta}{2} \end{array} \Bigg\} \frac{e^{-\theta}\sin 2\theta}{2} + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$

$\int e^{-\theta}\sin 2\theta\,d\theta \rightarrow \begin{array}{ll} u=e^{-\theta} & dv=\sin 2\theta\,d\theta \\ du=-e^{-\theta}d\theta & v=-\frac{\cos 2\theta}{2} \end{array} \longrightarrow$ (ii) $\boxed{\frac{e^{-\theta}\sin 2\theta}{2} - \frac{e^{-\theta}\cos 2\theta}{4} - \frac{1}{4}\int e^{-\theta}\cos 2\theta\,d\theta}$

Equating (i) and (ii), we get: $\int e^{-\theta}\cos\theta = \frac{2e^{-\theta}\sin 2\theta}{5} - \frac{e^{-\theta}\cos 2\theta}{5} + C$

18-) $\int_1^4 t^{1/2}\ln t\,dt \rightarrow \begin{array}{ll} u=\ln t & dv=t^{1/2}dt \\ du=\frac{1}{t}dt & v=\frac{2}{3}t^{3/2} \end{array} \Bigg\} \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \frac{2}{3}\int_1^4 t^{1/2}dt = \frac{2}{3}\sqrt{64}\ln 4 - \frac{2}{3}\cdot\frac{2}{3}t^{3/2}\Big|_1^4$

$= \frac{16}{3}\ln 4 - \frac{4}{9}(8-1) = \frac{16}{3}\ln 4 - \frac{28}{9}$

22-) $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr \rightarrow$ Using substitution rule:

$a=4+r^2$
$\frac{da}{2}=r\,dr$
$\rightarrow \underbrace{\int_4^5 \frac{a-4}{2\sqrt{a}}da}$; $r^2=a-4$

$r=1 \rightarrow a=4+1=5$
$r=0 \rightarrow a=4$

$\frac{1}{2}\left(\int_4^5 a^{1/2}da - \int_4^5 4a^{-1/2}da\right) = \frac{1}{2}\cdot\frac{2}{3}a^{3/2}\Big|_4^5 - 2\cdot 2a^{1/2}\Big|_4^5 = \frac{1}{3}(\sqrt{125}-\sqrt{64}) - 4(\sqrt{5}-\sqrt{4}) = \frac{16}{3} - \frac{7\sqrt{5}}{3}$

26-) $\int x^5\cos(x^3)dx \rightarrow$ Using substitution rule, then integ. by parts:

$a=x^3$
$da=3x^2dx \rightarrow \frac{da}{3}=x^2dx$
$\rightarrow \frac{1}{3}\int a\cos a\,da \Big\} \begin{array}{ll} u=a & dv=\cos a\,da \\ du=da & v=\sin a \end{array} \rightarrow \frac{1}{3}\left(a\sin a - \int \sin a\,da\right) = \frac{1}{3}a\sin a + \cos a + C$

$= \boxed{\frac{1}{3}x^3\sin x^3 + \cos x^3 + C}$

28-) $\int_1^4 e^{\sqrt{x}}\,dx$

$x=4 \to a=(4)^{\frac{1}{2}}=2$
$x=1 \to a=1$

$a = x^{\frac{1}{2}}$

$da = \frac{1}{2}x^{-\frac{1}{2}}dx \to 2a\,da = dx$

$$\to 2\int_1^2 a e^a\,da \left.\begin{matrix} u=a \\ du=da \end{matrix}\right\} \quad \begin{matrix} dv = e^a da \\ v = e^a \end{matrix} \quad \to 2ae^a\Big|_1^2 - 2\int_1^2 e^a da = 4e^2 - 2e^1 - (2e^2 - 2e^1) = 2e^2$$

34-)(a) If you had no problem with ex. 6, then this problem should be trivial;

$$\int \cos^n x\,dx = \int \cos^{n-1}x \cos x\,dx$$

$$\left.\begin{matrix} u=\cos^{n-1}x & dv = \cos x\,dx \\ du=(n-1)(\cos x)^{n-2}(-\sin x)dx & v=\sin x \end{matrix}\right\} \sin x\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,\overbrace{(\sin^2 x)}^{1-\cos^2 x}dx = \sin x\cos^{n-1}x + \int (n-1)(\cos^{n-2}x - \cos^n x)dx$$

$$\int \cos^n x\,dx = (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx + \sin x\cos^{n-1}x$$

$$\int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = (n-1)\int \cos^{n-2}x\,dx + \sin x\cos^{n-1}x$$

$$\boxed{\int \cos^n x\,dx = \frac{n-1}{n}\int \cos^{n-2}x\,dx + \frac{1}{n}\sin x\cos^{n-1}x}$$

$n=2$:

(b) $\int \cos^2 x\,dx = \frac{1}{2}\int dx + \frac{1}{2}\overbrace{\sin x\cos x}^{\frac{\sin 2x}{2}} = \frac{1}{2}x + \frac{1}{4}\sin 2x + C_0$

(c-) $n=4$

$$\int \cos^4 x\,dx = \frac{3}{4}\int \cos^2 x\,dx + \frac{1}{4}\sin x\cos^3 x = \frac{3}{4}\left(\frac{1}{4}\sin 2x + \frac{1}{2}x + C_0\right) + \frac{1}{4}\sin x\cos^3 x$$

$$= \frac{3}{16}\sin 2x + \frac{3}{8}x + \frac{1}{4}\sin x\cos^3 x + C$$

36-) From 35-) $\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx,\ n\geq 2$, so

$$\boxed{\int_0^{\pi/2}\sin^{2n}x\,dx = \frac{2n-1}{2n}\int_0^{\pi/2}\sin^{2n-2}x\,dx}$$

$n=1 \to \int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(\sin x)^0 dx = \frac{1}{2}\int_0^{\pi/2}dx = \frac{1}{2}\cdot\frac{\pi}{2}$

$n=2 \to \int_0^{\pi/2}\sin^4 x\,dx = \frac{3}{4}\int_0^{\pi/2}\sin^2 x\,dx = \left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\frac{3}{4}$

$n=3 \to \int_0^{\pi/2}\sin^6 x\,dx = \frac{5}{6}\int_0^{\pi/2}\sin^4 x\,dx = \frac{5}{6}\cdot\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$

A more elegant way of fully proving this problem is by "Mathematical Induction".

40-) (i) $\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$

Using (i), $\int x^4 e^x dx = x^4 e^x - 4\int x^3 e^x dx = x^4 - 4(x^3e^x - 3\int x^2 e^x dx) = e^x x^4 - 4(x^3 e^x - 3(x^2 e^x - 2\int x e^x dx)$

$= e^x x^4 - 4(x^3 e^x - 3(x^2 e^x - 2(xe^x - \int e^x dx)$

$= e^x (x^4 - 4x^3 + 12x^2 - 24x - 24) + C$

44-)

a-) $\int f(x)dx \Big\}$ $u = f(x)$ $dv = dx$

$du = f'(x)dx$ $v = x$ $\rightarrow xf(x) - \int x f'(x) dx$

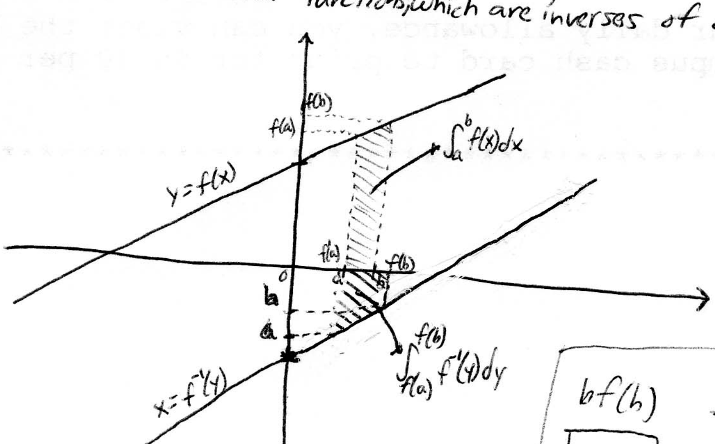
b-) Using pt. a, $\int_a^b f(x)dx = xf(x)\Big|_a^b - \int_a^b x \cdot f'(x)dx = bf(b) - af(a) - \underbrace{\int_a^b x f'(x) dx}_{\text{Using substitution}}$

: $y = f(x) \rightarrow \boxed{x = f^{-1}(y)}$ $\rightarrow$ $x = b \rightarrow y = f(b)$, $x = a \rightarrow y = f(a)$

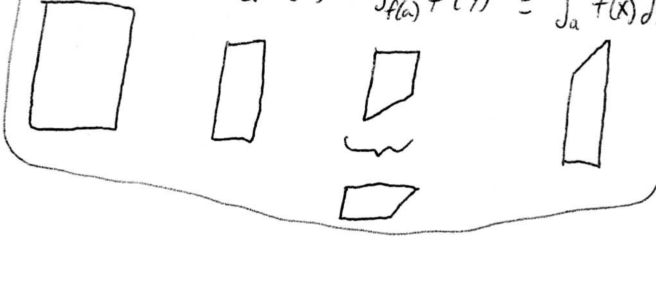
$\boxed{dy = f'(x)dx}$ pg. 64 (def. 2)

Now, $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} \overbrace{f^{-1}(y)}^{g(y)} dy$

c-) It'll be easier to draw two linear functions, which are inverses of one another



$bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) = \int_a^b f(x)d$



d-) Using pt. b

$\int_1^e \ln x\, dx = x\ln x\Big|_1^e - \int_1^e x \cdot \frac{1}{x} dx = e\overbrace{\ln e}^{1} - \overbrace{\ln 1}^{0} - x\Big|_1^e = e - e + 1 = 1$