Practice Final Exam
MAT 125

Fall 2014

The actual exam will consist of eight problems

1. Compute the following limits. Please distinguish between "$\lim f(x) = \infty$", "$\lim f(x) = -\infty$" and "limit does not exist even allowing for infinite values".

   (a) $\lim_{x\to -1} x^2 + x - 1$

   (b) $\lim_{x\to -3} \dfrac{x^2+2x-3}{x+3}$

   (c) $\lim_{t\to 0} \dfrac{\sqrt{2-t}-\sqrt{2}}{t}$

   (d) $\lim_{x\to 0} x \sin\left(\pi x^2 + \dfrac{\pi}{x^2}\right)$

   (e) $\lim_{x\to \infty} \dfrac{x^3+2x+1}{x^3-2x+1}$

   (f) $\lim_{x\to \pi/2} \dfrac{\cos x}{2x-\pi}$

   (g) $\lim_{x\to 0} \dfrac{\ln(\cos x)}{\cos(x)}$

   (h) $\lim_{x\to 0} \dfrac{\ln(\cos x)}{\sin(x)}$

   (i) $\lim_{x\to +\infty} (x - \ln(x))$

   (j) $\lim_{x\to 0^+} \sin(x)\ln(x)$

   (k) $\lim_{x\to 0} \left(\dfrac{\cos(x)}{\cos(2x)}\right)^x$

   (l) $\lim_{x\to 0^+} x^{1/x}$

   (m) $\lim_{x\to 0^+} \tan(x)^{5x}$

   (n) $\lim_{x\to +\infty} \left(\dfrac{1+x}{2+x}\right)^x$

2. Compute the derivatives of the following functions

   (a) $f(x) = x^3 - 12x^2 + x + 2\pi$

   (b) $f(x) = (2x+1)\sin(x)$

   (c) $g(s) = \sqrt{1+e^{2s}}$

(d) $h(t) = \frac{1+e^t}{1-e^t}$

(e) $f(x) = (2x+2)^{10}$

(f) $g(x) = x^{(\sin x)}$

(g) $\tan^{-1}\left(\frac{y}{1-y^2}\right)$

(h) $\sin^{-1}\left(\frac{1}{t^4}\right)$

3. Follow the scheme

   (a) Find asymptotes of $f(x)$

   (b) Determine intercepts

   (c) Symmetry: is the function even? odd?

   (d) Compute the derivative of $f(x)$

   (e) On which intervals is $f(x)$ increasing? decreasing?

   (f) What are local maxima and minima?

   (g) Find inflection points. On which intervals is $f(x)$ is concave up? down?

   (h) Sketch a graph of $f(x)$ using the results of the previous parts

   to sketch the graph of

   (a) $xe^{(-x^2)}$

   (b) $\frac{x+2}{x-2}$

   (c) $\frac{x^2}{x^2-3}$

4. Let $f(x) = \frac{1}{\sqrt{1+x}}$. Write the linear approximation for $f(x)$ near $x = 0$ and use it to estimate $f(0.1)$.

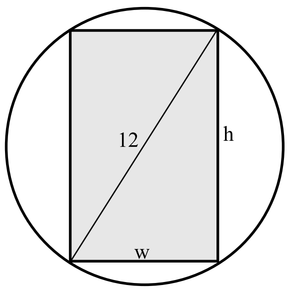
5. Use two iterations of Newton's method to determine an approximation to the root of $x = e^{-x^2}$. You are free to choose the initial point.

6. (a) It is known that the polynomial $f(x) = x^3 - x - 1$ has a unique real root. Between which two whole numbers does this root lie? Justify your answer.

   (b) Give an example of a function for which conditions of intermediate value theorem are not satisfied.

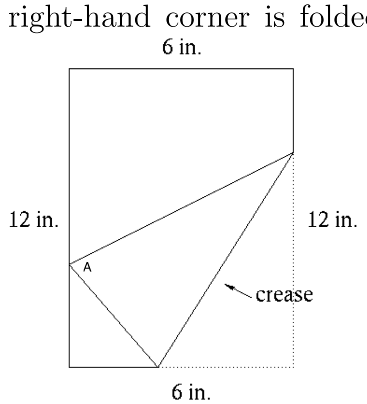
7. (a) It is known that for a rectangular beam of fixed length, its strength is proportional to $w \cdot h^2$, where w is the width and h is the height of the beam's cross-section.

   Find the dimensions of the strongest beam that can be cut from a 12" diameter log (thus, the cross-section must be a rectangle with diagonal 12").



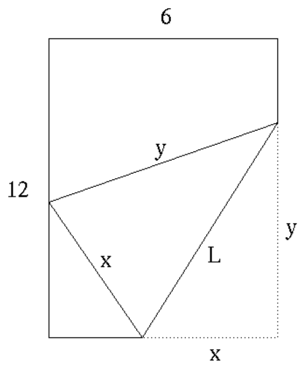
   (b) Find two nonnegative numbers whose sum is 9 and so that the product of one number with the square of the other number is maximized

   (c) A rectangular piece of paper is 12 inches high and six inches wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper

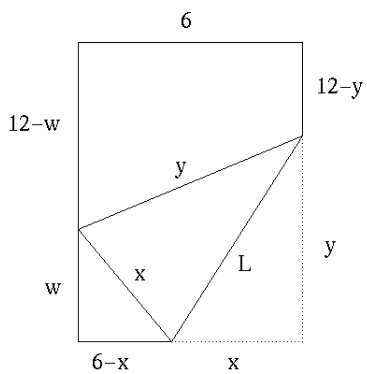


   Find the minimum length of the resulting crease. A is the right angle.

Solution of the last problem Let variable L represent the length of the crease and let variables x and y be as shown in the diagram.



(a) In this part of the solution we will write L as a function of x . Introduce variable w as shown in the following diagram.



It follows from the Pythagorean Theorem that

$$w^2 + (6 - x)^2 = x^2$$

so that

$$w^2 = x^2 - (x^2 - 12x + 36) = 12x - 36 \text{ and } w = \sqrt{12x - 36}.$$

Find a relationship between x and y . The total area of the paper can be computed from the areas of three right triangles $\triangle_1 = \triangle_{w,x,6-x}$, $\triangle_2$, $\triangle_3$, two of which $\triangle_2$, $\triangle_3$ are exactly the same dimensions, and one rhombus $\diamond = \diamond_{12-w,6,12-y,y}$. In particular

$$72 = (\text{total area of paper})$$

$= (\text{area of small triangle } \triangle_1) + 2(\text{area of large triangle } \triangle_2) + (\text{area of rhombus } \diamond)$

$= (1/2)(\text{length of base})(\text{height}) + 2(1/2)(\text{length of base})(\text{height}) + (\text{average height})(\text{length of}$

$$= (1/2)(6 - x)(\sqrt{12x - 36}) + (x)(y) + (1/2)\{(12 - y) + (12 - \sqrt{12x - 36}\}(6)$$

$$= (3 - (1/2)x)\sqrt{12x - 36} + xy + 3\{24 - y - \sqrt{12x - 36}\}$$

$$= 3\sqrt{12x - 36} - (1/2)x\sqrt{12x - 36} + xy + 72 - 3y - 3\sqrt{12x - 36}$$

$$= -(1/2)x\sqrt{12x - 36} + xy + 72 - 3y$$

i.e.,

$$72 = -(1/2)x\sqrt{12x-36} + xy + 72 - 3y$$

Solve this equation for y . Then

$$0 = -(1/2)x\sqrt{12x-36} + (x-3)y$$

$$(x-3)y = (1/2)x\sqrt{12x-36}$$

and

$$y = \frac{x\sqrt{12x-36}}{2(x-3)}$$

We wish to MINIMIZE the LENGTH of the crease

$$L = \sqrt{x^2+y^2}$$

Before we differentiate, rewrite the right-hand side as a function of x only. Then

$$L = \sqrt{x^2+y^2} = \sqrt{x^2 + \left(\frac{x\sqrt{12x-36}}{2(x-3)}\right)^2}$$

$$= \sqrt{x^2 + \frac{x^2(12x-36)}{4(x-3)^2}} = \sqrt{x^2 + \frac{x^2 12(x-3)}{4(x-3)^2}} = \sqrt{x^2 + \frac{3x^2}{x-3}}$$

(b) In this part we will minimize $L(x)$. We differentiate

$$L(x) = \sqrt{x^2 + \frac{3x^2}{x-3}}$$

using the chain rule and quotient rule, getting

$$L' = (1/2)\left(x^2 + \frac{3x^2}{x-3}\right)^{-1/2}\left\{2x + \frac{(x-3)(6x)-(3x^2)(1)}{(x-3)^2}\right\}$$

$$= \frac{2x + \frac{3x^2-18x}{(x-3)^2}}{2\sqrt{x^2+\frac{3x^2}{x-3}}} = \frac{2x + \frac{x(3x-18)}{(x-3)^2}}{2\sqrt{x^2+\frac{3x^2}{x-3}}}$$

(Factor out x from the numerator.)

$$= \frac{x\left(2 + \frac{3x-18}{(x-3)^2}\right)}{2\sqrt{x^2+\frac{3x^2}{x-3}}}$$

so that

$$x\left(2+\frac{3x-18}{(x-3)^2}\right)=0$$

Thus,

$x=0$ or $2+\dfrac{3x-18}{(x-3)^2}=0$,

$$-2=\frac{3x-18}{(x-3)^2}\Rightarrow -2(x-3)^2=3x-18\Rightarrow -2(x^2-6x+9)=3x-18\Rightarrow$$

$$-2x^2+12x-18=3x-18\Rightarrow -2x^2+9x=0\Rightarrow x(-2x+9)=0,$$

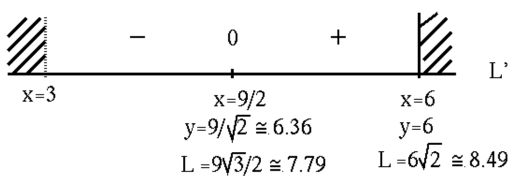
so that

$$x=0 \text{ or } (-2x+9)=0$$

, i.e.,

$$x=9/2.$$

Note that since the paper is 6 inches wide, it follows that $3<x\leq 6$. See the adjoining sign chart for L' .



If $x=9/2$ in. and $y=9/\sqrt{2}$ in. ≈ 6.36 in. , then $L=9\sqrt{3}/2$ in. ≈ 7.79 in. is the length of the shortest possible crease.

8. (a) The curve defined by the equation

$$y^2(y^2-4)=x^2(x^2-5)$$

is known as the "devil's curve". Use implicit differentiation to find the equation of the tangent line to the curve at the point $(0;-2)$.

(b) Repeat as above for

$$\sin y+x^2+4y=\cos x+\frac{\pi^2}{9}-\frac{1}{2}$$

at $(\frac{\pi}{3},0)$