(1) Let M be a fixed Riemannian manifold, and let $X : \mathbf{R} \to M$ be a map from the t-line $\mathbf{R}$ (endoowed with the metric $(dt)^2$) to M.

Let

$$L = \frac{1}{2} \int dt g_{IJ} \frac{dX^I}{dt} \frac{dX^j}{dt}. \tag{1}$$

(a) Describe the space W of critical points of L. (You should find that it is closely related to the space of geodesics on M.)

(b) Show that given any choice of a point $t_0 \in \mathbf{R}$, W has a natural identification with T^*M. (Use the metric on M to identify TM and T^*M.)

(c) Compute the symplectic structure ω on W. Show that for any choice of t_0, W becomes identified (under the identification in (b)) with T^*M with its natural symplectic structure.

(c) The vector field d/dt on $\mathbf{R}$ induces a vector field v on W. The "Hamiltonian" is a function H on W such that

$$dH = i_v \omega. \tag{2}$$

This is usually expressed by saying that "H generates time translations via Poisson brackets." Compute H.

(2) Consider $\mathbf{R}^{1,1}$ with even and odd coordinates t and θ and an odd distribution $\mathcal{A}$ generated by the vector field

$$D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}. \tag{3}$$

Thus, $D^2 = -\partial/\partial t$.

As in the last exercise, let M be a fixed Riemannian manifold. Let X be a map from $\mathbf{R}^{1,1}$ to M. Consider the Lagrangian

$$L = \frac{1}{2} \int_{\mathbf{R}^{1,1}} g_{IJ} \frac{\partial X^I}{\partial t} D X^j. \tag{4}$$

(a) Make sense of the definition of L by interpreting the integrand as a section of the Berezinian of the tangent bundle of $\mathbf{R}^{1,1}$.

(b) Setting $X^I = x^I + \theta \psi^I$, write L explicitly as an ordinary integral, over the t-line $\mathbf{R}^{1,0}$, of a function of x^I and ψ^I.

(c) Describe the space Y of critical points of L. You should get a description of the following kind. Describe the reduced space Y_{red} as an ordinary manifold, and the normal bundle to Y_{red} in Y as ΠV, where V is a vector bundle over Y_{red}. What are the "initial data" at a given point $t = t_0$ in $\mathbf{R}^{1,1}$ needed to determine a critical point? (This is the analog of problem 1(b) above.)

(d) Compute the symplectic structure on Y.

(e) The vector fields

$$\frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial t} \tag{5}$$

induce vector fields on Y. Find the functions Q and H that generate these vector fields by Poisson brackets. Explain why $\{Q, Q\} = 2H$.

Witten's Problems, Set One — N°. 2 (solution written by P. Deligne & D. Freed)

Preface.

The space of maps from $R^{1,1}$ to M is some kind of infinite dimensional supermanifold $\mathcal{F}$. We will not try to define it as a topological space provided with a sheaf of $\mathbb{R}$-algebras. We will only define it as a functor: for B a supermanifold, a map from B to $\mathcal{F}$ is by definition a morphism $R^{1,1} \times B \to M$. It is also called a B-point of $\mathcal{F}$, and it is viewed as a family of points parametrized by B. In the exercise, the phrase "let X be a map from $R^{1,1}$ to M" is an abuse of language for "let X be a map from $\mathbb{R}^{1,1}$ to M, depending on some parameter $b \in B$, for some super space B", i.e. "let B be a supermanifold and let X be a map from $R^{1,1} \times B$ to M".

This kind of interpretation of what a space of maps is does not apply only in the superworld. If we were to consider (as in exercise 1) the space of maps from $\mathbb{R}$ to M, one could try to put on it a structure of infinite dimensional manifold but, for the kind of problem we are considering, this is extra baggage. What is important is to know, for B a space "of parameters" (now an ordinary manifold), what is a family of maps $\mathbb{R} \to M$ parametrized by B.

"Working in components" means reinterpreting the functor $B \mapsto B$-points of $\mathcal{F}$ as follows.

A morphism $X\colon \mathbb{R}^{1,1} \times B \to M$ gives us

(a) a restriction $x\colon \mathbb{R} \times B \to M$. We view here $\mathbb{R}$ as the subvariety $\theta = 0$ of $\mathbb{R}^{1,1}$,

(b) an odd section $\psi := \frac{\partial}{\partial\theta} X\big|_{\mathbb{R}\times B}$ of the pullback by x of the tangent bundle T of M.

The construction $X \mapsto (x, \psi)$ is a bijection. Note that x^*T is an "even vector bundle", in the sense that it is of rank $(*, 0)$. This does not prevent it from having odd sections,for instance the product of the pullback of a section of T with an odd function on B.

This will usually be expressed with the abuse of language consisting in omitting B: a basis B is tacitly assumed to always be there, and one will say that to give a morphism X from $\mathbb{R}^{1,1}$ to M is the same thing as to give its restriction x to $\mathbb{R}$ and its derivative $\psi = \frac{\partial}{\partial\theta} X$ on $\mathbb{R}$ which is an odd section of x^*T. Taken literally, this statement would be useless: with no B tacitly assumed, the even vector bundle x^*T would have no nonzero odd section. When a B is present, one should beware that to give $x\colon \mathbb{R} \times B \to M$ is not just giving a morphism from $\mathbb{R} \times B_{\mathrm{red}}$ to M.

The lagrangian density is a map from $\mathcal{F} := \underline{\underline{\mathrm{Hom}}}(\mathbb{R}^{1,1}, M)$ to the space of densities on $\mathbb{R}^{1,1}$. This is to be interpreted in its functorial meaning. A family of densities on $\mathbb{R}^{1,1}$, parametrized by B, is by definition a section on $\mathbb{R}^{1,1} \times B$ of the pullback of the line bundle of densities on $\mathbb{R}^{1,1}$. The lagrangian density attaches, functorially in B to any family of maps from $\mathbb{R}^{1,1}$ to M parametrized by B, a family of densities on $\mathbb{R}^{1,1}$ parametrized by B.

"Families of densities" are also called "relative densities". An integration map is defined, which to a relative density on $V \times B$ over B with support proper over B, attaches a function on B. If V is a supermanifold of dimension (p,q), the line bundle of relative densities is of dimension $(1,0)$ or $(0,1)$, depending on the parity of q, and the integration map is even. The spaces of critical points of the lagrangian is a subspace of the space of all maps from $\mathbb{R}^{1,1}$ to M. As such, it is first to be defined as a functor: one has to define what it means for a map $X\colon B \to \underline{\underline{\mathrm{Hom}}}(\mathbb{R}^{1,1}, M)$, i.e. $\mathbb{R}^{1,1} \times B \to M$, to be a map to the subspace S of $\underline{\underline{\mathrm{Hom}}}(\mathbb{R}^{1,1}, M)$. The following two definitions are equivalent, with the second being closer to what is done in computations.

(i) For any supermanifold V, pointed by $o \in V$, and any $X_V\colon B \times V \to \underline{\underline{\mathrm{Hom}}}(\mathbb{R}^{1,1}, M)$, such that (a) it extends X: X is X_V restricted to $B \times 0$, (b) outside a compact region of $\mathbb{R}^{1,1}$, X_V coincide with X (X_V is a family of deformations of X, with compact support, parametrized by V), the following holds. Consider $\mathcal{L}(X_V) - \mathcal{L}(X)$, where $\mathcal{L}(X)$ is abuse of language for "pullback of $\mathcal{L}(X)$ by $\mathbb{R}^{1,1} \times B \times V \to \mathbb{R}^{1,1} \times B$. It is a relative density on $\mathbb{R}^{1,1} \times B \times V$ with support proper over $B \times V$. Integrate it, to get a function $\Delta \int \mathcal{L}$ on $B \times V$. One requires that the derivative of this function in the V-direction be zero on $B \times \{0\} \subset B \times V$.

(ii) In the second definition, one requires that after any change of basis $B' \to B$, (i) holds for $V = (\mathbb{R}, 0)$.

This defines the space of stationary points as a functor. In the exercise at hand, and assuming M to be complete, this functor is representable: the space of stationary points "is" an ordinary supermanifold of dimension $(2\dim M, \dim M)$, as we will see.

Solution. (a) The coordinates (t,θ) on $\mathbb{R}^{1,1}$ define a density $[t,\theta]$, also written $dt\,d\theta^{-1}$. The lagrangian density considered is

$$\mathcal{L} = -\tfrac{1}{2}\,dt\,d\theta^{-1}(\dot{X}, DX). \tag{2.1}$$

The minus sign is to make (2.5) below reasonable.

(b) If we choose local coordinates $\tilde{x}^i$ on M, a map $X\colon \mathbb{R}^{1,1} \to M$ can be written

$$X = (x^i(t) + \theta\psi^i(t)),$$

meaning $X^*\tilde{x}^i = x^i(t) + \theta\psi^i(t)$ with $x^i(t)$ (resp. $\psi^i(t)$) an even (resp. odd) function on $\mathbb{R}$, pulled back to $\mathbb{R}^{1,1}$ by $(t,\theta) \mapsto t$. Independently of the local coordinates, x is a map from $\mathbb{R}$ to M and ψ is an odd section of x^*T, for T the tangent bundle of M. The meaning of this, and in particular of "odd section of x^*T", was explained in the preface. The philosophy is that we tacitly work over a basis B, functorially in B. The map x is the restriction of X to the subvariety $\mathbb{R}$ of $\mathbb{R}^{1,1}$ defined by $\theta = 0$, while ψ is the restriction of $\partial_\theta X$ to $\mathbb{R}$. On $\mathbb{R}$, the vector fields ∂_θ and D are equal, and it will often be more convenient to describe ψ as the restriction of DX to $\mathbb{R}$. We will compute the density $\mathcal{L}$ on $\mathbb{R}^{1,1}$ in term of (x,ψ).

In local coordinates, $\mathcal{L}$ is $-dt\,d\theta^{-1}$ times

$$\tfrac{1}{2}X^*(g_{ij})\partial_t(X^*(\tilde{x}^i))D(X^*(\tilde{x}^j)). \tag{2.2}$$

As $\theta^2 = 0$, $X^*(g_{ij})$ is given by a finite Taylor series

$$X^*(g_{ij}) = g_{ij}(x(t) + \theta\psi(t)) = g_{ij} + \theta\psi^k\partial_k g_{ij}$$

with g_{ij}, $\partial_k g_{ij}$ and ψ^k evaluated at $x(t)$ and t. Expanding (2.2) gives

$$\begin{aligned} &\tfrac{1}{2}(g_{ij} + \theta\psi^k\partial_k g_{ij})(\dot{x}^i + \theta\dot{\psi}^i)(\psi^j - \theta\dot{x}^j) \\ &\qquad = \tfrac{1}{2}g_{ij}\dot{x}^i\psi^j + \tfrac{1}{2}\theta(g_{ij}\dot{\psi}^i\psi^j + \partial_k g_{ij}\dot{x}^i\psi^k\psi^j - g_{ij}\dot{x}^i\dot{x}^j). \end{aligned} \tag{2.3}$$

As $\psi^k\psi^j = -\psi^j\psi^k$, we have$\partial_k g_{ij}\psi^k\psi^j = -\partial_j g_{ik}\psi^k\psi^j$. As $g_{ij} = g_{ji}$, we similarly have $\partial_i g_{jk}\psi^k\psi^j = 0$, and

$$\partial_k g_{ij}\psi^k\psi^j = \tfrac{1}{2}(\partial_k g_{ij} - \partial_j g_{ik} + \partial_i g_{jk}) = g_{\ell j}\Gamma^\ell_{ik},$$

where the Γ are the Christoffel symbols defining the Levi-Civita connection. In (2.3), this allows us to rewrite

$$g_{ij}\dot{\psi}^i + \partial_k g_{ij}\dot{x}^i\psi^k = g_{ij}\dot{\psi}^i + g_{\ell j}\Gamma^{\ell}_{ik}\dot{x}^i\psi^k = g_{ij}(\nabla_t\psi)^i,$$

giving for (2.2)

$$\begin{aligned}(2.4)\qquad & \tfrac{1}{2}g_{ij}\dot{x}^i\psi^j + \tfrac{1}{2}\theta(g_{ij}(\nabla_t\psi)^i\psi^j - g_{ij}\dot{x}^i\dot{x}^j) \\ &= \tfrac{1}{2}(\dot{x},\psi) + \tfrac{1}{2}\theta((\nabla_t\psi,\psi) - (\dot{x},\dot{x})).\end{aligned}$$

Here is a shortcut from (2.3) to (2.4). Write (2.2) in the form $f(t) + \theta g(t)$. We have to compute $f(t_0)$ and $g(t_0)$ for each t_0. We now remember that we always work over a space B of parameters. The point t_0 is in fact a section of $\mathbb{R} \times B \to B$. The local coordinates $\tilde{x}_i$ used can be taken to vary with the parameters. This makes it possible to choose them so that at $x(t_0)$, i.e. along the section $x(t_0)$ of $M \times B \to B$, the metric tensor g_{ij} has vanishing first derivatives. The Christoffel symbols then vanish at $x(t_0)$, and (2.4) at t_0 results immediately from (2.3).

Here is a coordinate free derivation of (2.4). We use that for any function F on $\mathbb{R}^{1,1}$, if we write $F(t,\theta) = f_1(t) + \theta f_2(t)$, we have

$$f_1 = F \text{ restricted to } \mathbb{R}$$
$$f_2 = DF \text{ restricted to } \mathbb{R}\ .$$

We have $\psi = DX$ restricted to $\mathbb{R}$, so that for $F = (\dot{X}, DX)$, f_1 is $(\dot{x},\psi)$, while f_2 is the restriction to $\mathbb{R}$ of

$$D(\dot{X}, DX) = (\nabla_D\dot{X}, DX) + (\dot{X}, \nabla_D DX).$$

The Levi-Civita connection is torsion free, and D and ∂_t commute, while $D^2 = \frac{1}{2}[D,D] = -\partial_t$. It follows that

$$\nabla_D\dot{X} = \nabla_t DX$$
$$\nabla_D DX = \tfrac{1}{2}(\nabla_D DX + \nabla_D DX) = -\partial_t X$$

and $D(\dot{X}, DX)$, restricted to $\mathbb{R}$, is

$$(\nabla_t \psi, \psi) + (\dot{X}, -\dot{X}).$$

This agrees with (2.4).

Let $\mathfrak{L}'$ be the density on $\mathbb{R}$ deduced from $\mathfrak{L}$ by integrating along the fibres of the projection from $\mathbb{R}^{1,1}$ to $\mathbb{R}$ given by $(t, \theta) \mapsto t$. From (2.4), we get

$$\mathfrak{L}' = \tfrac{1}{2}(\dot{x}, \dot{x})dt - \tfrac{1}{2}(\nabla_t \psi, \psi)dt \tag{2.5}$$

The space of critical points is the same for $\mathfrak{L}$ and $\mathfrak{L}'$: it is the space of $X = (x, \psi)$ such that, for any deformation with compact support $X(u)$ of X, one has $\int \partial_u \mathfrak{L} = 0$ (resp. $\int \partial_u \mathfrak{L}' = 0$) at $u = 0$, and the density $\partial_u \mathfrak{L}'$ on $\mathbb{R}$ is deduced by θ-integration from the density $\partial_u \mathfrak{L}$ on $\mathbb{R}^{1,1}$.

(c) The computation of the Euler-Lagrange equations is parallel to the computations in problem 1. If X depends on an additional parameter u, and if δ stands for ∂_u or ∇_u, one has

$$\begin{aligned}\delta(\dot{X}, DX) &= (\delta\dot{X}, DX) + (\dot{X}, \delta DX) = (\nabla_t \delta X, DX) + (\dot{X}, \nabla_D \delta X)\\ &= \partial_t(\delta X, DX) - (\delta X, \nabla_t DX) + D(\dot{X}, \delta X) - (\nabla_D \dot{X}, \delta X).\end{aligned}$$

As $\nabla_t DX = \nabla_D \dot{X}$, this gives

$$\delta\mathfrak{L} = -dt\, d\theta^{-1} \left[-(\delta X, \nabla_t DX) + \tfrac{1}{2}(\partial_t(\delta X, DX) + D(\dot{X}, \delta X)) \right]. \tag{2.6}$$

The vector fields ∂_t and D are divergence free: the corresponding Lie derivative kills $dt\, d\theta^{-1}$. It follows that $dt\, d\theta^{-1}$ multiplied by the second term in [ ] of (2.6) is an exact differential (see (d) below) and the Euler-Lagrange equation is

$$\nabla_t DX = 0. \tag{2.7}$$

In the computation leading to (2.6), we took the additional parameter u, and hence δX, to be even. Why this suffices is explained in Deligne's appendix "Even rules" to Bernstein's lecture. Basically, an odd δX can be replaced by $\varepsilon \delta X$, for ε a new odd parameter.

To express (2.7) in term of the components (x, ψ) of X, one restricts $\nabla_t DX$ and $\nabla_D \nabla_t DX$ to $\mathbb{R} \subset \mathbb{R}^{1,1}$ $(\theta = 0)$. As $\mathbb{R} \subset \mathbb{R}^{1,1}$ is stable by ∂_t, the first gives

$$\nabla_t \psi = 0. \tag{2.8}$$

For the second, permuting ∇_D and ∇_t introduces a curvature term, while $\nabla_D DX = -\partial_t X$: one obtains

$$R(\psi, \dot{x})\psi - \nabla_t \dot{x} = 0 \tag{2.9}$$

Another method to obtain the Euler-Lagrange equations (2.8) (2.9) is to start from the lagrangian $\mathfrak{L}'$ (2.5).

Writing again δ for ∂_u or ∇_u, we have

$$\delta \mathfrak{L}' = (\delta \dot{x}, \dot{x})dt - \tfrac{1}{2}((\delta \nabla_t \psi, \psi) + (\nabla_t \psi, \delta \psi))dt$$

Permuting δ and ∇_t introduces a curvature term:

$$\delta \nabla_t \psi = R(\delta X, \dot{x})\psi + \nabla_t \delta \psi.$$

Integrating by part:

$$(\nabla_t \delta \psi, \psi) = \partial_t(\delta \psi, \psi) - (\delta \psi, \nabla_t \psi)$$
$$(\delta \dot{x}, \dot{x}) = (\nabla_t \delta x, \dot{x}) = \partial_t(\delta x, \dot{x}) - (\delta x, \nabla_t \dot{x})$$

and observing that $(\delta \psi, \nabla_t \psi) = -(\nabla_t \psi, \delta \psi)$, we obtain

$$\delta \mathfrak{L}' = -\left[(\nabla_t \psi, \delta \psi) + \tfrac{1}{2}(R(\delta x, \dot{x})\psi, \psi) + (\delta x, \nabla_t \dot{x})\right] dt + d\left((\delta x, \dot{x}) - \tfrac{1}{2}(\delta \psi, \psi)\right)$$

The Bianchi identity tells that if in $(R(\delta x, \dot{x})\psi, \psi)$ we cyclically permute δx, ψ and ψ, taking parities into account, the sum is zero:

$$(R(\delta x, \dot{x})\psi, \psi) - (R(\psi, \dot{x})\delta x, \psi) + (R(\psi, \dot{x})\psi, \delta x), \quad \text{hence}$$
$$(R(\delta x, \dot{x})\psi, \psi) = -2(R(\psi, \dot{x})\psi, \delta x).$$

We obtain

$$\delta\mathcal{L}' = -\left[(\nabla_t\psi, \delta\psi) + (\nabla_t\dot{x} - R(\psi, \dot{x})\psi, \delta x)\right]dt + d\left((\delta x, \dot{x}) - \tfrac{1}{2}(\delta\psi, \psi)\right), \tag{2.10}$$

giving again (2.8) and (2.9) as Euler-Lagrange equations.

The space Y of critical points of $\mathcal{L}$ or $\mathcal{L}'$ is hence the space of solutions (x, ψ) of (2.8), (2.9). The reduced space Y_{red} is obtained by imposing $\psi = 0$: It is the space found in problem 1, the space of solutions $x(t)$ of $\nabla_t\dot{x} = 0$.

The normal bundle of Y_{red} in Y is found by linearizing (2.8), (2.9) around $\psi = 0$: it is the odd vector bundle whose (even) sections are the odd sections ψ of x^*T obeying $\nabla_t = 0$.

The Cauchy data for (2.8) (2.9) are the data at some t_0 of $x(t_0)$, $\dot{x}(t_0)$ and $\psi(t_0)$. In term of X: X, $\dot{X}$ and DX at $(t_0, 0)$. If M is complete, Y maps isomorphically to the space of Cauchy data. This gives a description of Y as the vector bundle $T \times \Pi T$ over M (viewed as a space). In particular, we get a description of Y as an odd vector bundle over Y_{red}. Warning: this structure, and even the corresponding map $Y \to Y_{\text{red}}$, depend on the choice of t_0.

(d) The 2-form ω on the space Y of critical points can be computed using $\mathcal{L}$ or $\mathcal{L}'$. We first use $\mathcal{L}'$. The general recipe is to fix t_0, to extract from (2.10) the 1-form

$$\alpha_{t_0} = (\delta x, \dot{x}) - \tfrac{1}{2}(\delta\psi, \psi) \quad (\text{at } t_0) \tag{2.11}$$

on Y and to take $\omega = d\alpha_{t_0}$.

Let us identify Y with the space of Cauchy data at $t_0 = 0$: a point of Y becomes a triple $(x, \dot{x}, \psi)$ with x a point of M, $\dot{x}$ an (even) tangent vector at x and ψ an (odd) tangent vector at M. If we identify the tangent bundle with the cotangent bundle using the riemannian metric, Y becomes the pull back to T^*M of the odd tangent bundle of M. The term $(\delta x, \dot{x})$ in (2.11) becomes the pull back to Y of the canonical 1-form pdq on T^*M, and contributes to ω the pull back of the symplectic form $dp\,dq$ of T^*M.

The second term in (2.11) makes sense whenever on a manifold X (here T^*M) one considers an odd vector bundle ΠT, for T an orthogonal vector bundle with connection (here the pull back of the tangent bundle of M). It is the 1-form whose pull back by any section ψ (an odd section of T) is $-\frac{1}{2}(\nabla\psi, \psi)$. Its exterior derivative is the 2-form ω_T whose pull back by any ψ is given in local coordinates by

$$\begin{aligned}\psi^*(\omega_T)_{i,j} &= -\tfrac{1}{2}\partial_i(\nabla_j\psi, \psi) + \tfrac{1}{2}\partial_j(\nabla_i\psi, \psi)\\ &= -\tfrac{1}{2}(\nabla_i\nabla_j\psi, \psi) - \tfrac{1}{2}(\nabla_j\psi, \nabla_i\psi)\\ &\quad + \tfrac{1}{2}(\nabla_j\nabla_i\psi, \psi) + \tfrac{1}{2}(\nabla_i\psi, \nabla_j\psi)\\ &= -\tfrac{1}{2}(R_{ij}\psi, \psi) + (\nabla_i\psi, \nabla_j\psi)\end{aligned}$$

for R the curvature 2-form (with values in endomorphisms of T), i.e.

$$\psi^*\omega_T = \tfrac{1}{2}(\nabla\psi, \nabla\psi) - \tfrac{1}{2}(R\psi, \psi)$$

where the first term combines inner product (in T) and exterior product of differential forms. Final result:

$$\omega = \omega_{T^*M} + \tfrac{1}{2}(\nabla\psi, \psi) - \tfrac{1}{2}(R\psi, \psi). \tag{2.12}$$

To compute in terms of $\mathcal{L}$, one should start with the second term in (2.6) being

$$d\left[\tfrac{1}{2}i_t(-dt\, d\theta^{-1}(\delta X, DX)) + \tfrac{1}{2}i_D(dt\, d\theta^{-1}(\dot{X}, \delta X))\right].$$

In [ ], we have a 1-form A on the space of X, with values in integral codimension 1-forms on $\mathbb{R}^{1,1}$. The general recipe is to fix a space-like hypersurface Γ, for instance $t = t_0$, to integrate A on Γ to get a 1-form α_Γ on Y, and to take $\omega = d\alpha_\Gamma$.

The restriction of $i_X(\dots)$ to Γ depends only on the component of X normal to Γ. If Γ is $t = t_0$, we have

$$i_t(dt\, d\theta^{-1})\big|_\Gamma = d\theta^{-1} \quad \text{and} \quad i_D(dt\, d\theta^{-1})\big|_\Gamma = -\theta i_t(dt\, d\theta^{-1})\big|_\Gamma,$$

giving

$$A\big|_\Gamma = -\tfrac{1}{2}d\theta^{-1}(\delta X, DX) - \tfrac{1}{2}\theta d\theta^{-1}(\dot{X}, \delta X).$$

The integral of $d\theta^{-1} f$ on Γ is simply $\partial_\theta f$, or Df, evaluated at $(t_0, 0)$:

$$\int_\Gamma A = -\tfrac{1}{2} D(\delta X, DX) + \tfrac{1}{2}(\dot{X}, \delta X) \quad \text{at } (t_0, 0)$$
$$= -\tfrac{1}{2}(\nabla_D \delta X, DX) - \tfrac{1}{2}(\delta X, \nabla_D DX) + \tfrac{1}{2}(\dot{X}, \delta X) \quad \text{at } (t_0, 0)$$

We have $\nabla_D \delta X = \delta \psi$ and $\nabla_D DX = -\dot{x}$, at $(t_0, 0)$, so that (not surprisingly)

(2.13) $$\alpha_\Gamma \quad \text{is given by (2.11).}$$

(e) The space $\mathbb{R}^{1,1}$ is a group for the group law

$$(t, \theta) * (t', \theta') = (t + t' + \theta\theta', \theta + \theta').$$

Let us compute the bracket in the Lie algebra Lie$(\mathbb{R}^{1,1})$, identified with the tangent space to $\mathbb{R}^{1,1}$ at $(0,0)$. To ∂_θ at $(0,0)$ correspond over the odd dual numbers $\mathbb{R}[\varepsilon]$ the "infinitesimal" element $(0, \varepsilon)$, whose left translate by (t, θ) is $(t + \theta\varepsilon, \theta + \varepsilon) = (t - \varepsilon\theta, \theta + \varepsilon)$, corresponding to the left invariant vector field $\partial_\theta - \theta\partial_t = D$. Similarly, ∂_t at $(0,0)$ is the value at $(0,0)$ of the left invariant vector field ∂_t. On $\mathbb{R}^{1,1}$, $D^2 = -\partial_t$. It follows that

(2.14) $$\text{in Lie}(\mathbb{R}^{1,1}),\ \partial_\theta^2 = -\partial_t.$$

On $\mathbb{R}^{1,1}$, the vector fields D, ∂_t and the density $dt\, d\theta^{-1}$ are invariant by the group of left translations. Being built from them, so is $X \mapsto \mathcal{L}(X)$.

For a while, let us consider more generally maps $X \colon V \to M$ and a lagrangian density $\mathcal{L}(X)$ on V, with $X \mapsto \mathcal{L}(X)$ invariant by a group G acting on V. We also suppose given a G-stable cohomology class of "space like" hypersurfaces Γ, and assume that the corresponding 2-form ω on the space Y of extremals is non degenerate. To each Γ corresponds a 1-form α_Γ on Y, with $\omega = d\alpha_\Gamma$ and

(2.15) $$\alpha_\Delta - \alpha_\Gamma = d \int_\Gamma^\Delta \mathcal{L}.$$

By transport of structures, the group G acts on $\underline{\mathrm{Hom}}(V, M)$, respects Y and ω, and $g_*(\alpha_\Gamma) = \alpha_{g\Gamma}$. For $\tau \in \mathrm{Lie}(G)$, we will write $[\tau]$ for the vector field

$$\partial_\tau X(gv) \qquad \text{at} \qquad X(v)$$

on $\underline{\mathrm{Hom}}(V,M)$. The derivation is taken in g, at $g=e$. The action of G on $\underline{\mathrm{Hom}}(V,M)$ is $g\colon X\mapsto X(g^{-1}v)$; it induces $\tau\mapsto -[\tau]$ from $\mathrm{Lie}(G)$ to vector fields on $\underline{\mathrm{Hom}}(V,M)$. For a function on $\underline{\mathrm{Hom}}(V,M)$, one has

$$\partial_\tau(gF)=\partial_\tau(F(g^{-1}X))=[\tau]F\ ,\qquad \text{and} \tag{2.16}$$

$$\tau\longmapsto[\tau]\quad\text{is compatible with brackets.} \tag{2.17}$$

For $\tau\in\mathrm{Lie}(G)$, we now compute a generating function $T(\tau)$ for the symplectic vector field $[\tau]$ on E: $-dT(\tau)=i_{[\tau]}\omega$. One has $\omega=d\alpha_\Gamma$, hence

$$i_{[\tau]}\omega=\mathcal{L}_{[\tau]}\alpha_\Gamma-di_{[\tau]}\alpha_\Gamma.$$

The Lie derivative is

$$\mathcal{L}_{[\tau]}\alpha_\Gamma=\partial_\tau g\alpha_\Gamma=\partial_\tau\alpha_{g\Gamma}$$

with ∂_τ being a derivative in g at $g=e$. By (2.15), this equals $\partial_\tau d\int_\Gamma^{g\Gamma}\mathcal{L}$, and we take

$$T(\tau)=i_{[\tau]}\alpha_\Gamma-\partial_\tau\int_\Gamma^{g\Gamma}\mathcal{L}. \tag{2.18}$$

It is independent of the choice of Γ:

$$\begin{aligned}\left(i_{[\tau]}\alpha_\Delta-\partial_\tau\int_\Delta^{g\Delta}\mathcal{L}\right)&-\left(i_{[\tau]}\alpha_\Gamma-\partial_\tau\int_\Gamma^{g\Gamma}\mathcal{L}\right)\\&=i_{[\tau]}d\int_\Gamma^\Delta\mathcal{L}-\partial_\tau\int_{g\Gamma}^{g\Delta}\mathcal{L}=[\tau]\int_\Gamma^\Delta\mathcal{L}-\partial_\tau\int_{g\Gamma}^{g\Delta}\mathcal{L},\end{aligned}$$

and

$$[\tau]\int_\Gamma^\Delta\mathcal{L}(X)=\partial_\tau\int_\Gamma^\Delta\mathcal{L}(X(gv))=\partial_\tau\int_{g\Gamma}^{g\Delta}\mathcal{L}(X).$$

It follows from (2.17) that, up to a constant,

$$T([\tau_1,\tau_2])=\{T(\tau_1),T(\tau_2)\}=[\tau_1]T(\tau_2). \tag{2.19}$$

In fact, (2.19) holds exactly: as T is independent of Γ, we have by transport of structures

$$gT(\tau_2)=T(ad\,g(\tau_2)). \tag{2.20}$$

By (2.16) and the linearity of $T(\tau)$ in τ, (2.19) follows from (2.20) by applying the derivative ∂_τ in g.

We now take $V = \mathbb{R}^{1,1}$, with the supergroup $G = \mathbb{R}^{1,1}$ acting by left translations. The infinitesimal left translation (right invariant vector field) $\partial_\tau(gv)$ in $\mathbb{R}^{1,1}$ is

$$\begin{aligned} &D^+ = \partial_\theta + \theta\partial_t && \text{for} \quad \tau = (\partial_\theta \text{ at } (0,0)), \\ &\partial_t && \text{for} \quad \tau = (\partial_t \text{ at } (0,0)) \end{aligned}$$

and the vector field $[\tau]$ on $\underline{\mathrm{Hom}}(\mathbb{R}^{1,1}, V)$ is, respectively, D^+X at X and $\partial_t X$ at X. The generating functions are given by (2.18): respectively

$$\begin{aligned} Q &= i_{[\partial_\theta]}\alpha_\Gamma - \partial_\eta \int_\Gamma^{(0,\eta)\Gamma} \mathfrak{L} \qquad \text{and} \\ H &= i_{[\partial_t]}\alpha_\Gamma - \partial_t \int_\Gamma^{(t,0)\gamma} \mathfrak{L}, \end{aligned} \tag{2.21}$$

with the derivation evaluated at η (resp. t) $= 0$. Take for Γ the hypersurface $t = 0$. Its transform by $(0,\eta)$ is the hypersurface $t - \eta\theta = 0$. Let Y be the function on $\mathbb{R}$ equal to 1 for $x < 0$ and to 0 for $x > 0$. We have

$$Y(t - \eta\theta) = Y(t) + \eta\theta\delta(t)$$

and it follows that at $\eta = 0$,

$$\begin{aligned} \partial_\eta \int_\Gamma^{(0,\eta)\Gamma} \mathfrak{L} &= \partial_\eta \int (Y(t-\eta\theta) - Y(t))\mathfrak{L} \\ &= \int \theta\delta(t)\mathfrak{L}. \end{aligned}$$

Similarly,

$$\partial_t \int_\Gamma^{(t,0)\Gamma} \mathfrak{L} = \int \delta(t)\mathfrak{L}.$$

If $\mathfrak{L} = dt\, d\theta^{-1} L$, as $\theta d\theta^{-1} = -d\theta^{-1}\theta$, we get

$$\begin{aligned} \partial_\eta \int_\Gamma^{(0,\eta)\Gamma} \mathfrak{L} &= -L(0,0) \\ \partial_t \int_\Gamma^{(t,0)\Gamma} \mathfrak{L} &= (\partial_\theta L)(0,0) = DL(0,0). \end{aligned} \tag{2.22}$$

We now apply this to our $\mathcal{L}$ (2.1). Let us identify the space of extremals Y to the space of Cauchy data $(x, \dot{x}, \psi)$ at $(0,0)$. The 1-form $\alpha = \alpha_\Gamma$ is given by (2.11): for a variation of $(x, \dot{x}, \psi)$, it is $(\delta x, \dot{x}) - \frac{1}{2}(\delta\psi, \psi)$, with $\delta\psi$ a covariant derivative. The vector field $[\tau]$ is given by

$$
\begin{aligned}
&\partial_\tau (in\, g) \text{ [Cauchy data for } X(g(t,\theta)] \\
&= \nabla_t [X(t,\theta), \delta_t X(t,\theta), DX(t,\theta)] \quad \text{at } (0,0)
\end{aligned}
$$

for X the extremal with the given Cauchy data. This gives

$$
\begin{array}{cccc}
 & \delta x & \delta \dot{x} & \delta\psi \\
[\partial_\theta] & \psi & \nabla_D \dot{X}\big|_{(0,0)} = 0 & \nabla_D DX\big|_{(0,0)} = -\dot{x} \\
[\partial_t] & \dot{x} & \nabla_t \dot{X}\big|_{(0,0)} & \nabla_t DX\big|_{(0,0)} = 0
\end{array}
$$

and $i_{[\tau]}\alpha$ is

for $[\partial_\theta]$: $(\psi, \dot{x}) - \frac{1}{2}(-\dot{x}, \psi)$

for $[\partial_t]$: $(\dot{x}, \dot{x})$.

From (2.21) and (2.22), we get

$$
\begin{aligned}
Q &= (\psi, \dot{x}) - \tfrac{1}{2}(-\dot{x}, \psi) - \tfrac{1}{2}(\dot{x}, \psi) = (\psi, \dot{x}) \\
H &= (\dot{x}, \dot{x}) - D\left(-\tfrac{1}{2}(\dot{X}, DX)\right)\big|_{(0,0)} = \tfrac{1}{2}(\dot{x}, \dot{x})
\end{aligned}
$$

as $\nabla_D \dot{X} = \nabla_t DX = 0$ and $\nabla_D DX = -\dot{x}$.

By (2.14), (2.17) and (2.19), one has

$$\{Q, Q\} = [D^+]Q = -H.$$

Verification: $[D^+](\psi, \dot{x}) = (-\dot{x}, \dot{x})$.