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THE MOMENT MAP AND EQUIVARIANT COHOMOLOGY

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§1. INTRODUCTION

THE PURPOSE of this note is to present a de Rham version of the localization theorems of equivariant cohomology, and to point out their relation to a recent result of Duistermaat and Heckman and also to a quite independent result of Witten. To a large extent all the material that we use has been around for some time, although equivariant cohomology is not perhaps familiar to analysts. Our contribution is therefore mainly an expository one linking together various points of view.

The paper of Duistermaat and Heckman[11] which was our initial stimulus concerns the moment map $f: M \to R^l$ for the action of a torus T^l on a compact symplectic manifold M. Their theorem asserts that the push-forward by f of the symplectic (or Liouville) measure on M is a *piece-wise polynomial* measure on R^l. An equivalent version is that the Fourier transform of this measure (which is the integral over M of $e^{-i\langle \xi, f\rangle}$) is *exactly* given by stationary phase approximation. For example when $l = 1$ (so that T^l is the circle S) and the fixed points of the action are isolated points P, we have the *exact formula*

$$\int_M e^{-itf}\frac{\omega^n}{n!} = \sum_P \frac{e^{-itf(P)}}{(it)^n\, e(P)}, \tag{1.1}$$

where ω is the symplectic 2-form on M and the $e(P)$ are certain integers attached to the infinitesimal action of S at P. This principle, that stationary-phase is exact when the "Hamiltonian" f comes from a circle action, is such an attractive result that it seemed to us to deserve further study.

Now in equivariant cohomology there are well-known "localization theorems" which enable many computations to be reduced to the fixed-point set of the group action. This leads in particular to a general *integration formula* (see 3.8), and (1.1) turns out to be just a special case of this. The corresponding notions in K-theory were already used by Atiyah–Segal[4] to relate index-theory computations to fixed-points, and (via the theory of characteristic classes) their formulae are then closely related to ones we will derive here.

In Section 2 we recall the definition and basic properties of equivariant cohomology theory, and then in Section 3 we explain the localization theorems, which lead to the integration formula (3.8). This is still formulated in cohomological terms; but in Section 4 we describe how, using a de Rham version, we end up with an integration formula for genuine differential forms. The symplectic case is fitted into this framework in Section 6, where we show that extending the symplectic form ω to an equivariant closed form $\omega^{\#}$ is equivalent to giving a moment map. The detailed deduction of the Duistermaat–Heckman result is then described in Section 7.

Section 5 is devoted to examining the relation of Witten's results to equivariant cohomology. As we shall see his work can be interpreted as giving a Hodge-theoretic proof of the localization theorem.

The use of equivariant de Rham theory also sheds light on an old paper of Bott[7] concerning the zeroes of Killing vector-fields. This is explained in an appropriate context in Section 8. The connection between Bott's paper and the Duistermaat–Heckman result was also noted independently by Berline and Vergne[5] and quite recently also by Duistermaat–Heckman[12].

There are a number of interesting cases of infinite-dimensional symplectic manifolds where the Duistermaat–Heckman principle appears to hold. These examples (some of which were pointed out to us by Witten) involve functional integrals of the type familiar in quantum field theory. None of the proofs of the integration formula apply directly in infinite dimensions because integration now involves some regularization procedure. It is therefore an extremely interesting and challenging problem to establish a suitable infinite-dimensional version of the Duistermaat–Heckman integration formula.

§2. THE EQUIVARIANT THEORY REVIEWED

The equivariant cohomology of a G-space M is defined as the ordinary cohomology of the space M_G obtained from a fixed universal G-bundle EG, by the mixing construction

$$M_G = EG \times_G M. \tag{2.1}$$

In (2.1) G acts on the right of EG and on the left on M, and the notation means that we identify $(pg, q) \sim (p, gq)$ for $p \in EG$, $q \in M$; $g \in G$. Thus M_G is the bundle with fibre M over the classifying space BG associated to the universal bundle $EG \to BG$; and π will denote the corresponding natural projection:

$$\begin{array}{c} M_G \\ \downarrow{\scriptstyle \pi} \\ BG. \end{array} \tag{2.2}$$

We also have a natural map σ of M_G onto the quotient space M/G, which fits into the mixing diagram of Cartan and Borel:

$$\begin{array}{ccccc} E & \longleftarrow & E \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ B & \underset{\pi}{\longleftarrow} & (E \times_G M) & \underset{\sigma}{\longrightarrow} & M/G. \end{array} \qquad \begin{array}{l} E = EG \\ B = BG. \end{array} \tag{2.3}$$

In general σ is *not* a fibering. In fact it is easy to see that the fiber $\sigma^{-1}(G\,m)$ over the orbit Gm is the quotient of E by the stabilizer G_m of m, so that $\sigma^{-1}(Gm)$ is the classifying space of G_m:

$$\sigma^{-1}(Gm) = EG/G_m \simeq BG_m. \tag{2.4}$$

Under reasonable assumptions on G and M, which are certainly satisfied if G is a compact Lie group acting smoothly on M, this map σ induces a *homotopy equivalence whenever G acts freely, so that $M_G \simeq M/G$ in that case.* On the other hand M_G is a "better" functorial construction and turns out to be the proper "homotopy theoretic quotient" of M by G. In any case, the equivariant cohomology, denoted by $H_G^*(M)$, is defined by

$$H_G^*(M) = H^*(M_G), \tag{2.5}$$

and constitutes a contravariant functor from G-spaces to modules over the base ring

$$H_G^* = H_G^*(pt) = H^*(BG). \tag{2.6}$$

This module structure is of course induced by the projection π, while the map σ defines a natural arrow

$$\sigma^*: H^*(M/G) \to H_G^*(M) \tag{2.7}$$

which is an *isomorphism* if G acts freely.

Note furthermore that in this notation (2.4) is expressed by

$$H_G^*(G/H) = H_H^* \tag{2.8}$$

for any closed subgroup H of G, and finally that there is a natural homomorphism

$$\imath^*: H_G^*(M) \to H^*(M) \tag{2.9}$$

from equivariant theory to ordinary theory which corresponds to the inclusion

$$i: M \to M_G$$

of M as the fiber over the basepoint of BG.

Remarks. (1) The functorial nature of the construction $M \mapsto M_G$ enables one to "extend" all the concepts of ordinary cohomology to the equivariant one in an essentially routine manner.

The term "extend" is also very appropriate in this context since ordinary data on M should be thought of as being given on the fiber over the basepoint of BG, whereas equivariant data extend these to all of M_G. For example if V is a vector-bundle over M, then any action of G on V (lifting the action of G on M) serves to define a vector-bundle V_G over M_G, which extends the original V to all of M_G. The characteristic classes of such a bundle then naturally take values in $H_G^*(M)$ and incorporate the lifting data. Of course truly geometric bundles come equipped with a natural lifting which we will then take for granted. An important case in point is the normal-bundle ν_m of a G-orbit through a point $m \in M$. In view of (2.8) its characteristic classes lie in $H_{G_m}^* = H^*(BG_m)$, where G_m is the stability group of m. In particular as we saw in §1 these classes can be non-trivial even if the orbit through m is just the point m itself, i.e. when m is a fixed point of the action.

(2) Although the equivariant theory is largely analogous to the ordinary theory, its overtones are rather different, precisely because the equivariant theory of a "point" is so large. The correct way to understand this phenomenon is to observe that in the equivariant theory the "points" of M/G really correspond to orbits G/H of G on M, and hence are of various "sizes" according to the size of H. Thus the free orbits carry only zero-dimensional equivariant cohomology, while the point orbits stand at the other extreme and contribute H_G^*.

For a compact connected group G, and working over the reals $\mathbb{R}$, the base ring is simply a polynomial ring:

$$H_G^* = \mathbb{R}[u_1, \ldots, u_l] \tag{2.10}$$

with l generators of even degree, where l is the rank (the dimension of a maximal torus $T \subset G$). Furthermore for a torus T these generators are all of dimension 2, and if T is a maximal torus of G the natural arrow

$$H_G^*(M) \xrightarrow{\simeq} H_T^*(M)^W \tag{2.11}$$

imbeds the l.h.s. onto the elements of $H_T^*(M)$ which are invariant under the group of automorphisms of T which are induced by inner automorphisms of G. Thus the well-known principle that the study of compact Lie groups reduces to the behavior of the Weyl group W acting on a maximal torus of G also applies to the equivariant theory, and explains our later preoccupation with the torus case.

We recall the salient points of the proof of (2.11). If one divides $EG \times M$ by T first, one obtains a fibering

$$G/T \to M_T \to M_G$$

with fiber G/T. Now the fact that the Euler class of G/T is non-zero already implies that, over $\mathbb{R}$, $H^*(M_G)$ imbeds into $H^*(M_T)$. Finally the identification of the image follows from the well-known fact that W acts on $H^*(G/T)$ as the regular representation.

We turn next to the equivariant form of the "umkehrungs" homomorphism

$$f_*: H^*(N) \to H^{*+q}(M), \qquad \dim M - \dim N = q$$

associated to maps $f: N \to M$ of compact oriented manifolds. Recall that in ordinary cohomology this "push forward" has the following properties.

It is functorial:

$$(f \circ g)_* = f_* \circ g_*. \tag{2.12}$$

It is a homomorphism of $H^*(M)$-modules: i.e.

$$f_*(vf^*u) = (f_*v)u. \tag{2.13}$$

If f is a fibering, f_* corresponds to integration over the fiber. (2.14)

When $f: N \hookrightarrow M$ is the inclusion of a submanifold then f_* factors through the Thom isomorphism: that is, in the diagram

$$\begin{array}{ccccc} H^{*-1}(M - N) & \xrightarrow{\delta} & H^*(M, M - N) & \xrightarrow{j^*} & H^*(M) \\ & & \wr\!\uparrow \Phi_N & & \\ & & H^{*-q}(N) & & \end{array} \tag{2.15}$$

we have

$$f_* = j^* \circ \Phi_N, \tag{2.17}$$

with Φ_N the Thom isomorphism.

Here of course $H^*(M, M - N)$ is, by excision, a purely N-local quantity, so that in the differentiable category this group can be identified with $H^*(\nu_N; \nu_N - N)$, where ν_N is the normal bundle to N in M, and thence finally with the compactly supported cohomology of ν_N:

$$H^*(M, M - N) \simeq H_c^*(\nu_N). \tag{2.18}$$

Recall finally that one calls $\Phi_N 1 \in H_c^q(\nu_N)$ the *Thom class* of the normal bundle and that its restriction to N is precisely the *Euler class* of ν_N.

Thus for the inclusion $f: N \hookrightarrow M$ *we have*

$$f^* f_* 1 = \mathrm{Euler}(\nu_N). \tag{2.19}$$

In accordance with our first remark this push-forward is now seen to extend word-for-word to the equivariant situation. To verify this recall that by the graph construction every map can be factored into an inclusion followed by a fibering (and in fact product) projection. Hence it is sufficient to check the properties (2.14) and (2.15) in the equivariant theory. Now when f is a fibering, the induced map $f_G: N_G \to M_G$ is one also, and integration over the fiber is well defined in any fibering with an oriented compact manifold as fiber. Similarly the *usual* Thom isomorphism, but now applied to bundles over M_G, extends the classical one to the equivariant theory.

It should also be clear from the foregoing that this equivariant push forward—denoted by f_*^G—preserves the H_G^*-module structure, and finally that the push forward π_*^G of the map

$$M \xrightarrow{\pi} pt$$

corresponds to integration over the fiber in the fibering $M_G \to BG$.

§3. THE LOCALIZATION THEOREM

The main thing which distinguishes equivariant from ordinary cohomology is that it has a much larger coefficient ring, namely the polynomial ring H_G^*. The extra structure which this gives is in turn related to the orbit-structure of the G-action. In this section we shall review the main results in this direction. These results go back to the early work of P. A. Smith and more especially to A. Borel who pioneered the use of the spaces M_G in the theory of transformation groups. A more formal approach, utilizing concepts from algebra, was systematically explored in Quillen[15] while the corresponding ideas in equivariant K-theory were used in connection with the index theorem by Atiyah and Segal[4].

Before embarking on the general case it may be helpful to compare the simple case when G is the circle group S with that of ordinary *integral* cohomology. Since the integers Z and the polynomial ring in one variable $R[u]$ are both principal ideal domains the basic structure of Z-modules and $R[u]$-modules is very similar. In both cases a fintely generated module is the direct sum of a *free* module and a *torsion* module. As a crude approximation we may ignore the torsion module and concentrate on the free part which is determined by its *rank*, i.e. the number of free generators. This is frequently done in integral cohomology and is equivalent to working with rational cohomology. In the equivariant case one can go further and *identify the rank with the ordinary rank* (sum of Betti numbers) *of the fixed point set* F. This is the content of the "localization theorem" which we are going to explain.

Instead of ignoring all Z-torsion we can instead ignore only torsion involving a given set of primes, say those dividing some fixed integer f. In the same way for $R[u]$-modules we can ignore torsion arising from a given polynomial $f(u)$. This will give more precise results.

The "primes" in $R[u]$ are the irreducible polynomials, and it is therefore a little more convenient to replace R by the complex field C. The "primes" in $C[u]$ now just correspond to complex "points". From now on therefore we work over C, so that $H_G^*(M)$ will denote equivariant cohomology with complex coefficients. This involves no loss of information and it simplifies the presentation. It even leads to stronger conclusions.

When $G = T^l$ is a torus of dimension l, then H_G^* is the polynomial ring $C[u_1, \ldots, u_l]$, and this is no longer a principal ideal domain. As a consequence modules over this ring no longer have such a simple structure. However, one can still define the torsion sub-module and the rank. This can be done by passing to the field of fractions $C(u_1, \ldots, u_k)$. If we want more precise information we must use some of the notions of commutative algebra (see for example Atiyah–Macdonald[3]). The most important of these is that of the *support* of a (finitely generated) module. For a module H over $C[u_1, \ldots, u_l]$ its support is the subset of C^l defined by:

$$\operatorname{Supp} H = \bigcap_f V_f \text{ over all } f \in C[u_1, \ldots, u_l] \text{ with } fH = 0, \tag{3.1}$$

where V_f is the hypersurface $f(u) = 0$. Thus a free module has the whole space C^l as support, while the support of a torsion module indicates which "primes" occur in the torsion.

A related notion is that of "localization". For example if $f \in C[u_1, \ldots, u_l]$ is any non-zero polynomial, localizing to the open set $U_f = C^l - V_f$ means that we allow ourselves to divide by powers of f. More formally we form the ring $C[u_1, \ldots, u_l]_f$ consisting of all rational functions with denominator a power of f, and consider the corresponding module

$$H_f = H \otimes_{C[u_1, \ldots, u_l]} C[u_1, \ldots, u_l]_f \tag{3.2}$$

for any module H over $C[u_1, \ldots, u_l]$. This process kills torsion supported in V_f, i.e.

$$\operatorname{Supp} H \subset V_f \Rightarrow H_f = 0.$$

Used systematically, the notion of localization associates to any $C[u_1, \ldots, u_l]$-module H a *sheaf* $\mathscr{H}$ on C^l, and the support of H is just the support of this sheaf. Those familiar with sheaf theory should think in these terms, but since our use of these ideas is quite elementary we shall not insist on the sheaf-theory and will stick to modules.

For a *graded* module H over $C[u_1, \ldots, u_l]$, where $\deg u_i = 2$, we have a natural action of C^* given by:

$$\lambda(h) = \lambda^{2q} h \text{ for } h \in H^q.$$

This is compatible with the process of localization provided we make C^* also act on C^l in the obvious way: $\lambda(u) = \lambda^2 u$. This means that the sheaf $\mathscr{H}$ is also acted on by C^*. In particular this means that $\operatorname{Supp} H$ *is a cone*, i.e. it is C^*-invariant. Note that, for $l = 1$, this implies that *the torsion sub-module of H is supported at* 0, so that H becomes free when we localize to $C - 0$.

Since the variables u_i have degree 2 the graded module H is the direct sum of two sub-modules:

$$H = H^{\text{even}} \oplus H^{\text{odd}}.$$

These can be localized separately and, in particular, we can define their ranks. Note however that the integer graduation essentially gets lost on localization.

So far we have discussed the case when G is a torus. For a general compact connected

Lie group we know that

$$H_G^* \cong (H_T^*)^W,$$

where T is a maximal torus and W is the Weyl group. The support of an H_G^*-module can then be defined naturally as a W-invariant subset of C^l. However we shall not really be concerned with this more general case.

Returning to a torus T with

$$H_T^* = C[u_1, \ldots, u_l]$$

we recall that the variables u_i should be viewed naturally as coordinates on the Lie algebra t, or its complexification t^c. Thus the support of a module over this cohomology ring is naturally a subset of t^c.

After these preliminaries we now examine what information on the structure of the equivariant cohomology ring can be deduced from various assumptions on the action. As a first step we have the following

LEMMA (3.3). *If there is a T-map $X \to T/K$, where K is a closed subgroup of T, then*

$$\operatorname{Supp} H_T^*(X) \subset k^c.$$

Proof. The T-maps $X \to T/K \to$point give ring homomorphisms $H_T^*(X) \leftarrow H_T^*(T/K) \leftarrow H_T^*$. But

$$H_T^*(T/K) \cong H_K^* \cong H_{K_0}^*$$

where K_0 is the identity component of K (note that $K = K_0 \times$ finite group and we are using complex coefficients so that finite groups can be ignored). Thus $H_T^*(X)$ is effectively a module over $H_{K_0}^*$ which becomes a module over H_T^* by restriction from T to the sub-torus K_0. The lemma now follows from the naturality of supports.

Remarks. (1) If Y is the inverse image of a point in the map $X \to T/K$ then Y is a K-space and $X = T \times_K Y$ the extension to a T-space. Thus

$$H_T^*(X) \cong H_K^*(Y)$$

and our Lemma says that supports are compatible with "extension".

(2) The Lemma is only of interest when $K \neq T$, so that k^c is a proper subspace of t^c. In particular $H_T^*(X)$ is then a torsion module.

(3) The Lemma applies in particular when X is any orbit of a T-action with isotropy group K, and more generally when X is a tubular neighborhood of an orbit in a T-manifold M (such tubes being constructed as usual by use of a T-invariant metric on M).

(4) Applying the Mayer–Vietoris sequence and observing that, for any exact sequence of modules over $C[u_1, \ldots, u_l]$:

$$D \to E \to F$$

$$\operatorname{Supp} E \subset \operatorname{Supp} D \cup \operatorname{Supp} F,$$

Remark 3 leads by a simple induction to the following key result:

PROPOSITION (3.4). *Let T act smoothly on the compact manifold M with F as set of* *stationary points. Then*

$$\operatorname{Supp} H_T^*(M - F) \subset \bigcup_K k^c$$

where K runs over all isotropy groups. In particular $H_T^*(M - F)$ is a torsion module over H_T^*.

The same result holds for any T-invariant subspace Y of $M - F$ and hence, by exact sequences, for the equivariant relative cohomology of any pair in $M - F$. Apply this to the pair $(M - U, \partial(M - U))$ where U is an open tubular neighbourhood of F in M and we deduce the main localization theorem:

THEOREM (3.5). *The kernel and cokernel of*

$$i^*: H_T^*(M) \to H_T^*(F)$$

have support in $\bigcup_K k^c$, where K runs over the finite set of all isotropy groups $\neq T$. In particular both modules have the same rank.

Since $H_T^*(F) \cong H^*(F) \otimes H_T^*$ is a free module the last part of the theorem implies

COROLLARY (3.6). rank $H_T^*(M) = \dim H^*(F)$ *and the same holds for the individual odd and even parts.*

The theorem also implies that $H_T^*(M)$ becomes a free module once we localize to an open set $U_f \subset t^c$ *where f is any polynomial which vanishes on all k^c.*

These results take a specially simple form when $T = S$ is a circle. Then $k^c = 0$, so that $\operatorname{Ker} i^*$ and $\operatorname{Coker} i^*$ have support at $0 \in C$ and $H_S^*(M)$ becomes free on $C - 0$. Equivalently the torsion subgroup of $H_S^*(M)$ is annihilated by a power of u. As we saw earlier this follows directly from the fact that H is a *graded* module. In the general torus case however the grading simply tells us that the support of the torsion sub-module of $H_T^*(M)$ is a proper cone, while Theorem (3.5) is more precise.

If we use Proposition (3.4) in the exact sequence of the pair $(M, M - F)$ we can similarly deduce that for the push-forward

$$i_*: H_T^*(F) \to H_T^*(M)$$

both $\operatorname{Ker} i_*$ and $\operatorname{Coker} i_*$ are torsion modules (and more precisely are annihilated by some power of any polynomial f vanishing on all k^c). Combining this with Theorem (3.5) it follows that the composition $i^* i_*$ is an isomorphism modulo torsion. This can be seen more directly and explicitly as follows.

As explained in §2 i_* is a module homomorphism (over $H_T^*(M)$) and $i^* i_*$ is therefore multiplication by the equivariant Euler class of the normal bundle v_F

$$i^* i_* 1 = E(v_F).$$

It follows that $E(v_F)$ must be invertible in the localized module

$$H_T^*(F)_f = (H_T^*)_f \otimes H^*(F),$$

where f is a suitable polynomial. To see how this comes about let $F = \{P\}$ be the decomposition of F into its connected components, and we may consider separately each $E(v_P)$. Since terms of positive degree in $H^*(P)$ are nilpotent an element in $H^*(P)$ is invertible provided its component in $H^0(P)$ is nonzero. Hence $E(v_P)$ will be invertible in some localization $H_T^*(P)_f$ provided its component in $H_T^* \otimes H^0(P)$ is a non-zero polynomial (which can then be taken as f). But this component is determined by restricting to any point p of P. The action of T on the fibre v_p has no fixed vectors (because the only fixed directions are tangential to P), so that v_p decomposes as a direct sum of non-trivial 2-dimensional representations of T. These can be oriented (compatibly with chosen orientations of M and P) and so viewed as complex characters $\alpha_j: T \to U(1)$. If $\alpha_j = \exp(2\pi i a_j)$ so that

$$a_j = \sum a_{jk} u_k$$

is a linear form on the Lie algebra, the equivariant Euler class of v_p is just given by

$$E(v_p) = \prod_j a_j. \tag{3.7}$$

If we denote this polynomial by f_P it follows that $i^* i_*$ becomes invertible after localizing with respect to

$$f_F = \prod_P f_P.$$

Working over such a localized ring (or over the full field of rational functions) we therefore see that

$$Q = \sum_P \frac{i_P^*}{E(v_P)}$$

is inverse to $i_*^F: H_T^*(F) \to H_T^*(M)$. Thus for any $\phi \in H_T^*(M)$ we have (after localizing)

$$\phi = i_*^F Q\phi = \sum_P \frac{i_*^P i_P^*}{E(v_P)}.$$

Applying the push-forward to a point

$$\pi_*^M: H_T^*(M) \to H_T^*$$

to both sides, and using the functoriality of push-forwards we deduce the

Integration formula: $$\pi_*^M \phi = \sum_P \pi_*^P \left\{ \frac{i_P^* \phi}{E(v_P)} \right\}. \tag{3.8}$$

Remarks. (1) Using the de Rham model of $H_T^*(M)$ to be explained in §4, (3.8) will in fact lead to explicit formulae, replacing integration over M by integration over the components P of the stationary set of the action. In the context of symplectic geometry we shall see in §7 how it leads to the Duistermaat–Heckman formula.

(2) The corresponding formula also holds in equivariant K-theory and is the basis of the applications to the equivariant index theorem (Atiyah–Segal [4]). The Chern character gives a natural transformation $K \to H^*$ which explains the similarity between the formulae we are deriving here and those that occur in connection with the index theorem.

§4. THE EQUIVARIANT DE RHAM THEORY

A de Rham model for the equivariant theory goes back to the foundational work on characteristic classes of Cartan, Chern, Chevalley and Weil of the late forties and early fifties and we will here briefly review their constructions starting with Weil's de Rham model for the universal fibering $EG \to BG$.

Let then $\mathfrak{g}$ denote the Lie algebra of G, and write $W(\mathfrak{g})$ for the tensor-product:

$$W(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* \tag{4.1}$$

of the exterior algebra and the symmetric algebra on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. This "Weil algebra" is next graded by assigning dimension 1 to an element $\theta \in \mathfrak{g}^*$ in the exterior algebra, and degree 2 to the corresponding element—usually denoted by u—in the symmetric algebra. Thus if the $\{\theta^\alpha\}$ form a basis for $\mathfrak{g}^*$, then the Weil algebra is freely generated as *commutative graded algebra*, i.e. $\omega^p\omega^q = (-1)^{pq}\omega^q\omega^p$, by the generators θ^α and u_α of degree 1 and 2 respectively:

$$W(\mathfrak{g}) = \mathbb{R}[\theta; u]. \tag{4.2}$$

Geometrically the θ^α are of course to be interpreted as left invariant forms on the group G and, so understood, their exterior derivatives can be expanded in terms of (exterior) products of the θ^α

$$d\theta^\alpha + \frac{1}{2}\sum c^\alpha_{\beta\gamma}\theta^\beta\theta^\gamma = 0, \tag{4.3}$$

where the constants $c^\alpha_{\beta\gamma}$—skew in β, γ—are the *structure constants* of $\mathfrak{g}$ relative to the base θ^α.

In terms of these, $W(\mathfrak{g})$ is now endowed with a differential operator D, which on our generators is defined by:

$$\begin{aligned} D\theta^\alpha + \frac{1}{2}\sum c^\alpha_{\beta\gamma}\theta^\beta\theta^\gamma + u_\alpha &= 0, \\ Du_\alpha - \sum c^\alpha_{\beta\gamma}u_\beta\theta^\gamma &= 0, \end{aligned} \tag{4.4}$$

and then extended to all of $W(\mathfrak{g})$ as an antiderivation.

The Jacobi identity satisfied by the $c^\alpha_{\beta\gamma}$, which is equivalent in our context to the assertion that $d^2\theta^\alpha = 0$, is now seen to imply that $D^2 = 0$ in $W(\mathfrak{g})$. For instance when G reduces to a torus T, so that the $c^\alpha_{\beta\gamma}$ vanish, (4.3) reduces to

$$D\theta^\alpha + u_\alpha = 0; \quad Du_\alpha = 0$$

so that the D-cohomology of $W(\mathfrak{g})$—written $H_D\{W(\mathfrak{g})\}$—simply reduces to $\mathbb{R}$ in dimension 0. Now, as $W(\mathfrak{g})$ is to be a de Rham model for the contractible space EG, this it as it should be, and the first theorem of the subject is that quite generally:

$$H^*_D\{W(\mathfrak{g})\} = \mathbb{R}. \tag{4.5}$$

In a geometric context for a smooth principal bundle P with structure group G and base space M, the elements X of $\mathfrak{g}$ appear naturally as vertical vector-fields, corresponding to the infinitesimal right translations of P in the direction of X. Hence $X \in \mathfrak{g}$ acts naturally on the de Rham complex $\Omega^*(P)$ of differential forms on P, both by the inner product $i(X)$ and by the Lie-derivative $\mathfrak{L}(X)$. Furthermore these operations are linked by the fundamental "infinitesimal homotopy" identity:

$$\mathfrak{L}(X) = i(X)d + di(X), \tag{4.6}$$

where d is the exterior derivative in $\Omega^*(P)$.

Finally recall that under the projection $P \to M$, $\Omega^*(M)$ becomes identified with the *basic elements* of $\Omega^*(P)$, that is, the elements φ characterized by

$$i(X)\varphi = 0 \quad \mathfrak{L}(X)\varphi = 0 \quad \text{all } X \in \mathfrak{g}. \tag{4.7}$$

In the Weil algebra $W(\mathfrak{g})$ these operations of $X \in \mathfrak{g}$ are defined by:

$$\begin{aligned} &i(e_\alpha)\theta^\beta = \delta_\alpha{}^\beta; \quad i(e_\alpha)u_\beta = 0 \\ &\mathfrak{L}(e_\alpha) = i(e_\alpha)D + Di(e_\alpha) \end{aligned} \tag{4.8}$$

where $\{e_\alpha\}$ is a base for $\mathfrak{g}$ dual to our θ^α, and the basic subcomplex $B\mathfrak{g}$ of $W(\mathfrak{g})$—again defined by (4.7)—is then easily seen to reduce to the ring of *polynomials on* $\mathfrak{g}$, *invariant under the coadjoint action of* $\mathfrak{g}$ *on* $\mathfrak{g}^*$:

$$B\mathfrak{g} \simeq \operatorname*{Inv}_{\mathfrak{g}} S(\mathfrak{g}^*). \tag{4.9}$$

Now for a compact connected Lie group G, this ring of invariants is itself a polynomial ring in $l = \operatorname{rank} G$ generators, and serves as a de Rham model for $H^*(BG)$. That is, one has a natural isomorphism:

$$B\mathfrak{g} \xrightarrow{\simeq} H^*(BG). \tag{4.10}$$

For instance when G reduces to a torus T of rank l, $B\mathfrak{g}$ clearly reduces to $\mathbb{R}[u_1, \ldots, u_l]$, the polynomial ring in the u_i, in accordance with our earlier description of $H^*(BT)$.

Remarks. In general $B\mathfrak{g}$ is not an adequate model for BG even if G is connected. For instance if $G = \mathbb{R}$, BG has trivial cohomology, but $B\mathfrak{g} = \mathbb{R}[u]$, does not. On the other hand, if $G = SL(2)$ then $H^*(BG)$ is a polynomial ring on a generator of degree 2 while $B\mathfrak{g}$ is a polynomial ring on a generator of degree 4. For a more precise analysis of the relation of $H^*(BG)$ and $B\mathfrak{g}$ see for instance (Bott[9]).

The proof of the isomorphism (4.10) for compact groups, is based on three facts. First of all, that BG can be approximated up to any dimension by a geometric fibering (e.g. the fibering of S^{2n+1} over $\mathbb{C}P^n$ approximates the universal fibering $EG \to BG$, for G the circle, up to dimensions $2n$). Secondly, that the connection–curvature construction serves to define a natural homotopy class of G-maps from $W(\mathfrak{g})$ to $\Omega^*(P)$ for any geometric principal fiber-bundle P. Finally, that for a compact connected group G, the inclusion of the left invariant differential forms

$$\Lambda\mathfrak{g}^* \hookrightarrow \Omega^* G \tag{4.11}$$

into the de Rham complex of G induces an isomorphism in cohomology.

This fact, in conjunction with the acyclicity of $W(\mathfrak{g})$, then translates into the isomorphism $B\mathfrak{g} \simeq H^*(BG)$.

In view of the preceding it should now be clear how to construct an infinitesimal de Rham model—noted $\Omega_{\mathfrak{g}}^*(M)$—for $M_G = M \underset{G}{\times} EG$.

Namely, one *defines* $\Omega_{\mathfrak{g}}^*(M)$ by:

$$\Omega_{\mathfrak{g}}^*(M) = \text{basic complex of } \{\Omega^*(M) \otimes W(\mathfrak{g})\}. \tag{4.12}$$

With this understood the arguments sketched above are seen to extend essentially work-for-word to the following, also classical.

THEOREM 4.13. *If G is a compact connected group acting smoothly on the manifold M, there is a natural isomorphism*:

$$H\{\Omega_{\mathfrak{g}}^*(M)\} \simeq H_G^*(M).$$

It is instructive to analyze $\Omega_{\mathfrak{g}}^*(M)$ in greater detail when G reduces to a circle S. In that case $W(\mathfrak{g}) \simeq \mathbb{R}[\theta, u]$ so that every $\varphi \in \Omega^*(M) \otimes W(\mathfrak{g})$ has a unique decomposition into a finite sum:

$$\varphi = \sum_k a_k u^k + \sum_l b_l u^l \theta \tag{4.14}$$

$$k, l = 0, 1 \ldots \text{ and } a_k, b_l \in \Omega^*(M).$$

Hence if X denotes the generator of $\mathfrak{g} \simeq \mathbb{R}$, dual to θ, that is $i(X)\theta = 1$, a *basic* φ must satisfy the requirements

$$i(X)\varphi = \sum_k \{i(X)a_k\}u^k + \sum_l (b_l + i(X)b_l\theta)u^l = 0$$

and

$$\mathfrak{L}(X)\varphi = \sum_k \{\mathfrak{L}(X)a_k\}u^k + \sum_l \{\mathfrak{L}(X)b_l\}u^l = 0.$$

On the other hand in view of the independence of the u^k and $u^k\theta$ over $\Omega^*(M)$ these conditions are quite equivalent to:

$$\mathfrak{L}(X)a_k = 0; \quad b_k = -i(X)a_k \tag{4.15}$$

and can now be recast in the following more palatable form.

Let Ω_X^* *denote the kernel of* $\mathfrak{L}(X)$ *in* $\Omega^*(M)$. These are then the forms on M *invariant under our circle action*. Next let $\Omega_X^*[u]$ be the polynomial ring generated by a generator of degree 2 over Ω_X^*, and define a ring homomorphism

$$\Omega_X^*[u] \xrightarrow{\lambda} \Omega^*(M) \otimes W(\mathfrak{g})$$

by the formulae:

$$\begin{aligned} \lambda(a) &= a - i(X)a\theta \\ \lambda(u) &= u. \end{aligned} \tag{4.16}$$

The condition (4.15) then clearly implies that λ induces a ring isomorphism

$$\Omega_X[u] \simeq \Omega_{\mathfrak{g}}^*(M). \tag{4.17}$$

Thus $\Omega_X[u]$ inherits a differential operator d_X from the natural one in $\Omega_{\mathfrak{g}}^*(M)$, characterized by the condition

$$\lambda d_X = D\lambda. \tag{4.18}$$

Now

$$\begin{aligned} D\lambda a &= D(a - i(X)a\theta) \\ &= da - i(X)da\theta + i(X)au \quad \text{using (4.6) and (4.15)} \\ &= \lambda\{da + i(X)au\} \end{aligned}$$

whence we see that

$$d_X a = da + i(X)au. \tag{4.19}$$

Further, as u is closed in $\Omega_{\mathfrak{g}}^*(M)$, it follows that

$$d_X u = 0, \tag{4.20}$$

and these two conditions now uniquely determine the differential operator d_X on $\Omega_X[u] \simeq \Omega_{\mathfrak{g}}^*(M)$.

Remarks. A first consequence of (4.19) is of course that an invariant closed form $a \in \Omega^*(M)$, i.e. a form a in $\Omega_X(M)$, need *not determine* an *equivariantly closed* form. Indeed it does so if and only if in addition $i(X)a = 0$, i.e. if it *is basic relative to the action of X*. On the other hand even if $i(X)a \neq 0$ it may be possible to "extend" a to an equivariant form by adding to a some polynomial p in the ideal generated by u in $\Omega_X[u]$ so that $a' = a + P$ is d_X-closed.

In short the natural map $\Omega_X^*[u] \to \Omega^*(M)$ obtained by setting $u = 0$ plays the role of the restriction of forms from M_G to a fiber M. Similarly, if M is compact and oriented, the integration over the fiber π_* in M_G is represented in $\Omega_X^*[u]$ by ordinary integration over M. Thus

$$\pi_*(a_k u^k) = \left(\int_M a_k\right) u^k. \tag{4.21}$$

Secondly, note that this very tractable model for $\Omega_{\mathfrak{g}}^*(M)$ when G is a circle extends without difficulty to the torus case: $G = T$. Indeed all that has to be done to describe $\Omega_G^*(M)$ when T has rank l, is to choose a basis $X_1, \ldots, X_l$ in its Lie algebra, and then adjoin to the T-invariant forms of $\Omega^*(M)$—still denoted by $\Omega_X^* - l$ indeterminates $u_1, \ldots, u_l$, of degree 2, and to set

$$d_X = da + \sum_k i(X_k)au_k. \tag{4.22}$$

For the nonabelian case it is of course not possible to identify $\Omega_{\mathfrak{g}}^*(M)$ quite so simply, and it is usually expedient to describe this case in terms of the Weyl group invariant terms in $\Omega_T^*(M)$ where T is a maximal torus of G.

§5. RELATION WITH WITTEN'S COMPLEX

In the second part of (Witten[17]), Witten considers a Killing field (which we call X) on a compact Riemann manifold M, and introduces the operators

$$d_s = d + si(X), \tag{5.1}$$

where s is a real parameter, acting on the total de Rham complex $\Omega^*(M)$. He then studies the associated "Hamiltonian"

$$H_s = d_s d_s^* + d_s^* d_s \tag{5.2}$$

and proves that for $s \neq 0$ the dimension of its null-space (which we shall denote by W_s) is the sum of the Betti numbers of the set of zeroes of the Killing field X.

It is our purpose in this section to explain carefully how this result of Witten is related to the results we have described in the preceding sections. We shall also make a few comments comparing the methods of proof.

Witten first observes that W_s is contained in the space Ω_X^* of forms annihilated by the Lie derivative of $\mathfrak{L}(X)$. Now since $d_s^2 = s\mathfrak{L}(X)$ it follows that $d_s^2 = 0$ on Ω_X^* and hence, by standard Hodge theory arguments, W_s can be identified with the de Rham groups

$$H(\Omega_X^*, d_s) = \operatorname{Ker} d_s / \operatorname{Im} d_s.$$

This shows, as usual, that W_s is essentially independent of the metric (so long as X preserves it), so that it is a diffeomorphism invariant of the action. Finally, an easy rescaling argument shows that $W_s \cong W_t$ for s, t both non-zero.

Our first task is to explain how these Witten groups W_s are related to our equivariant cohomology groups. For simplicity we first assume that X generates a circle group S. Our equivariant cohomology $H_S^*(M)$ is then a module over a polynomial ring $C[u]$ (if we use complex coefficients) and we saw in §4 that

$$H_S^*(M) = H(\Omega_X^*[u], d_X)$$

where d_X is defined as in (5.1) but with u instead of s. Thus the only difference between Witten's groups and ours is that he considers s as a real non-zero *parameter* while we consider u as an *indeterminate*. But, in appropriate circumstances, which as we shall see are satisfied here, one may interchange evaluation with formation of homology: more precisely we have the following easy algebraic lemma:

LEMMA (5.3). *Let (C^*, d) be a cochain complex of free $C[u]$-modules and let s be a complex number not in the support of the torsion sub-module of $H(C^*, d)$. Then*

$$H(C_s^*, d_s) \cong H(C^*, d)/(u - s)H(C^*, d)$$

where on the left we have put $u = s$ before taking cohomology and on the right we have put $u = s$ after taking cohomology.

Proof. The assumption on s means that multiplication by $(u - s)$ on $H = H(C^*, d)$ is injective. Introduce now the usual exact sequences

$$0 \to Z \to C \xrightarrow{d} B \to 0$$

of cocycles Z and coboundaries B. If for any of these modules A we put $A_s = A/(u-s)A$ then, since B, Z and C are all free modules, we trivially get the exact sequence

$$0 \to Z_s \to C_s \to B_s \to 0.$$

The hypothesis on H_s is just what is needed to give the exactness of

$$0 \to B_s \to Z_s \to H_s \to 0,$$

as one easily verifies (and as will be proved more generally below). These two together yield the lemma.

Remark. This Lemma is the precise counterpart of the familiar fact that, if the usual integral cohomology H is free, then the mod p cohomology is just H mod p. In general the universal coefficient theorem gives a relation involving the torsion subgroup. In the same way our Lemma can be extended to deal with the torsion at $s = 0$ of $H_S^*(M)$. This leads to the formula

$$\dim H^*(M) = \dim H^*(F) + 2 \dim J \tag{5.4}$$

where J is the kernel of multiplication by $(u-s)$ on $H_S^*(M)$.

Now as we saw in §3 the fact that $H_S^*(M)$ is a *graded* module already implies that its torsion is supported only at 0. Hence for any $s \neq 0$ we have by Lemma (5.3) a natural isomorphism:

$$W_s \cong H_S^*(M)/(u-s)H_S^*(M). \tag{5.5}$$

Moreover $H_S^*(M)$ being free on $\mathbf{C} - 0$ shows that the vector space on the right has constant dimension independent of s (for $s \neq 0$), and this is also the rank of $H_S^*(M)$ as $C[u]$-module. This is consistent with Witten's observation that dim W_s is constant for $s \neq 0$.

Remarks. (1) In geometric or sheaf-theory terms we can say that W_s is the fibre of the vector bundle on $C - 0$ associated to the locally free sheaf $H_S^*(M)$.

(2) Witten's group can equally be defined for complex values of s, so there is no need to restrict to real values in what we have said.

The situation is very similar for the general case of a torus action except that u now stands for $(u_1, \ldots, u_l) \in C^l$. Equivariant cohomology is computed from $\Omega_X^*(M)$ using the operator d_X given in (4.22), namely

$$d_X = d + \sum_k u_k i(X_k).$$

Now the X_k are a basis for the Lie algebra of T so that we can write

$$X = \sum \xi_k X_k$$

and hence Witten's differential is

$$d_s = d + s \sum \xi_k i(X_k).$$

Thus d_s is obtained from d_X by putting $u_k = s\xi_k$, or just $u = sX$ if we regard both u and X as in t^c.

From the localization theorem of §3 we know that $H_T^*(M)$ is free over $t^c - \bigcup_K k^c$, where K runs over the isotropy groups $\neq T$. In particular $X \notin k$ otherwise it would not generate T. Hence

$$X \in t^c - \bigcup_K k^c$$

and $H_T^*(M)$ is free at all points sX with $s \neq 0$. To identify the Witten groups, as in (5.5), but with S now replaced by T we therefore need the appropriate generalization of Lemma (5.3). This is

LEMMA (5.6). *Let (C^*, d) be a cochain complex of free $C[u_1, \ldots, u_l]$-modules and assume that, for some polynomial f, $H(C^*, d)_f$ is a free module over the localized ring $C[u_1, \ldots, u_l]_f$. Then, if $s \in C^l$ with $f(s) \neq 0$,*

$$H(C_s^*, d_s) \cong H(C^*, d) \bmod m_s$$

where m_s is the ideal generated by $(u_1 - s_1, \ldots, u_l - s_l)$.

Proof. We shall require some homological algebra for our proof this time so we assume familiarity with the Tor functors. As a first step, since localization is an exact functor, we can replace the polynomial ring $C[u_1, \ldots, u_l]$ by $A = C[u_1, \ldots, u_l]_f$. Our assumption is then that $H = H(C^*, d)_f$ is a free A-module. As before we now consider the exact sequences of A-modules

$$0 \to Z^q \to C^q \xrightarrow{d} B^{q+1} \to 0$$

$$0 \to B^q \to Z^q \to H^q \to 0.$$

We now tensor these with A/m_s. Since H^q is free the second sequence gives an exact sequence

$$0 \to B_s^q \to Z_s^q \to H_s^q \to 0$$

where as before we write H_s for $H \otimes_A A/m_s$ etc. Also we get isomorphisms

$$\mathrm{Tor}_j^A(B^q, A/m_s) \cong \mathrm{Tor}_j^A(Z^q, A/m_s) \text{ for } j \geq 1.$$

From the first sequence on the other hand, since C^q is free, we get isomorphisms

$$\mathrm{Tor}_{j+1}^A(B^{q+1}, A/m_s) \cong \mathrm{Tor}_j^A(Z^q, A/m_s) \text{ for } j \geq 1.$$

Combining these and iterating we see that

$$\mathrm{Tor}_j^A(B^q, A/m_s) \cong \mathrm{Tor}_{j+n}^A(B^{q+n}, A/m_s) \text{ for } \begin{array}{l} j \geq 1 \\ n \geq 1. \end{array}$$

But the polynomial ring $C[u_1, \ldots, u_l]$ and all its localizations have homological dimension l so that all $\mathrm{Tor}_j^A = 0$ for $j > l$ (Cartan–Eilenberg[10]. Hence taking $n = l$ we see that

$$\mathrm{Tor}_j^A(B^q, A/m_s) = 0 \text{ for all } j \geq 1.$$

From the first of our exact sequences this then implies the exactness of

$$0 \to Z_s^q \to C_s^q \to B_s^{q+1} \to 0$$

and the lemma follows as before.

In view of (5.5), and its generalization to the general torus case, we see that Witten's result that

$$\dim W_s = \dim H^*(F) \tag{5.7}$$

is essentially equivalent to the Corollary of the Localization Theorem that

$$\operatorname{rank} H_T^*(M) = \dim H^*(F). \tag{5.8}$$

Our proof of (5.8) depended on the standard properties of cohomology (exactness, excision) applied in the equivariant context. Witten's method is totally different and quite novel in that he applies asymptotic expansions familiar in Quantum Mechanics to a cohomological problem. He works with the Hodge model of W_s, i.e. the null-space of the operator H_s in (5.2), and then considers the behaviour of H_s as $s \to \infty$. This has the effect of concentrating the relevant analysis around F and leads to the equality (5.7).

Note finally that Witten's rescaling argument corresponds to our use of the grading in $H_T^*(M)$. As we observed this implies that the corresponding sheaf on C^l is naturally acted on by C^*, so that the localizations at u and su will be isomorphic for $s \neq 0$.

§6 RELATIONS WITH THE MOMENT MAP

The moment map makes its appearance whenever a G-action on M preserves a *symplectic* form ω on M. Thus $\omega \in \Omega^2(M)$ is a closed form on M, with the property that the *symplectic volume* $v = \omega^n/n!$ is nowhere zero on M. This nondegeneracy of ω at every point enables one to associate to every 1-form θ a corresponding vector field $\boldsymbol{\theta}$, and dually to every vector field X a 1-form $\mathbf{X}$, by the assignment:

$$\theta = i(\boldsymbol{\theta})\omega, \qquad \mathbf{X} = i(X)\omega, \tag{6.1}$$

and in terms of this correspondence the *Poisson bracket* of two functions f and g on M is given by:

$$\{f, g\} = \mathbf{df}(g). \tag{6.2}$$

This operation is now seen to induce a Lie algebra structure $\mathfrak{L}_\omega M$ on the C^∞-functions on M. Furthermore if $\mathrm{Vect}_\omega(M)$ denotes the Lie algebra of vector fields on M which preserve ω, then the assignment

$$f \to \mathbf{df}$$

is also seen to define a natural extension of Lie algebras:

$$0 \to \mathbb{R} \to \mathfrak{F}_\omega M \to \mathrm{Vect}_\omega M \to 0. \tag{6.3}$$

In this framework consider now a symplectic action of a Lie algebra $\mathfrak{g}$ on (M, ω), that is, a Lie-homomorphism

$$\mu : \mathfrak{g} \to \mathrm{Vect}_\omega(M). \tag{6.4}$$

Any lifting $\tilde{\mu}$ of μ to a Lie-homomorphism

$$\tilde{\mu}:\mathfrak{g}\to\mathfrak{F}_{\omega}M \tag{6.5}$$

is called a "comoment" of μ or a *Poisson action* on (M, ω), and the corresponding *moment map* is then defined by duality in the following manner.

Given $\tilde{\mu}$ and a point $p \in M$, the evaluation of $\tilde{\mu}(X)$ at p is a linear functional $\Phi_p:\mathfrak{g}\to\mathbb{R}$. Hence as p ranges over M the Φ_p may be thought of as a map:

$$\Phi: M \to \mathfrak{g}^*. \tag{6.6}$$

This is the moment map associated to $\tilde{\mu}$, and it spreads M out over $\mathfrak{g}^*$ equivariantly, in the sense that all the vector fields $\mu(X)$ are projectable relative to Φ and project onto the natural "coadjoint" action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

Quite recently Weinstein pointed out that this construction and some of its applications were already known to Sophus Lie. In any case many remarkable properties of the moment mapping have been discovered in recent years by the modern school of symplectic geometers.

To obtain a more concrete understanding of the moment map, return to the example of a circle S acting on M and let X be an infinitesimal generator of the action. To lift this action to a Poisson action one has to assign to X a C^∞-function f_X on M, such that

$$\mathbf{d}f_X = X, \text{ or equivalently such that } df = \mathbf{X} \equiv i(X)\omega. \tag{6.7}$$

Thus the obstruction to finding such an f_X is simply the cohomology class of $i(X)\omega$ in $H^1(M)$.

If this class vanishes, f_X exists and is then unique up to a constant, i.e. an element of $H^0(M)$. Now in our equivariant context the function f_X has another interpretation. Consider $\omega \in \Omega^2(M)$. By assumption ω is invariant under S, i.e. annihilated by $\mathfrak{L}(X)$, and hence lies in $\Omega_X{}^2[u]$. It is now a natural question whether so interpreted ω can be extended to an *equivariantly closed* form. For dimensional reasons the only way to "extend" ω in $\Omega_X[u]$ is to add a multiple of u to ω. Thus all the extensions take the form

$$\omega^\# = \omega - fu \tag{6.8}$$

where f is some C^∞-function on M. Computing via (4.19) we find that

$$d_X\omega^\# = (i(X)\omega - df)u \tag{6.9}$$

so that (6.8) is an *equivariantly closed extension if and only if $f = f_X$ defines a Poisson lifting of the action.*

This phenomenon is quite general, as we will now demonstrate.

Suppose then that $X\mapsto f_X$ is a Poisson lifting of the G-action under consideration. In terms of a base $\{X_\alpha\}$ for $\mathfrak{g}$, we write $X_\alpha\mapsto f_\alpha$ for this assignment. Also let i_α stand for $i(X_\alpha)$ and for the rest use the notation of Section 4 with $\{\theta^\alpha\}\in\mathfrak{g}^*$ a dual base to $\{X_\alpha\}$ and $\{u_\alpha\}\in S^1(\mathfrak{g}^*)$ its "twin" in the symmetric part of $W(\mathfrak{g})$. With all this understood consider now the element $\omega^\# \in \Omega^*(M)\otimes W(\mathfrak{g})$ given by

$$\omega^\# = \omega - D\sum_\alpha f_\alpha\theta^\alpha.$$

Then we assert that $\omega^\#$ *is a basic and D-closed extension of ω to $\Omega^*_{\mathfrak{g}}(M)$.*

Proof. In our notation the Poisson conditions amount to:

$$i_\alpha \omega = \mathrm{d}f_\alpha \text{ and } X_\alpha f_\beta = f_{[X_\alpha, X_\beta]} = \sum_\gamma c^\gamma_{\alpha\beta} f_\gamma, \tag{6.11}$$

or equivalently—as $X_\alpha f_\beta = i_\alpha \, \mathrm{d}f_\beta$—to:

$$i_\alpha \omega = \mathrm{d}f_\alpha \text{ and } i_\alpha i_\beta \omega = \sum_\gamma c^\gamma_{\alpha\beta} f_\gamma. \tag{6.12}$$

Now expanding $\omega^{\#}$ we obtain

$$\omega^{\#} = \omega + \sum \mathrm{d}f_\alpha \theta^\alpha + \sum f_\alpha \, \mathrm{d}\theta^\alpha - \sum f_\alpha u^\alpha \tag{6.13}$$

so that

$$\begin{aligned} i_\beta \omega^{\#} &= i_\beta \omega + \sum i_\beta \, \mathrm{d}f_\alpha \theta^\alpha - \mathrm{d}f_\alpha + \sum f_\alpha i_\beta \, \mathrm{d}\theta^\alpha \\ &= 0. \end{aligned}$$

The terms on the right cancel by virtue of (6.12) and (4.3).

Thus $\omega^{\#}$ satisfies the first requirement of a basic element. Next observe that

$$D\omega^{\#} = \mathrm{d}\omega + D^2\left(\sum f_\alpha \theta^\alpha\right) = 0 \tag{6.14}$$

because $\mathrm{d}\omega = 0$ on M and $D^2 = 0$ in $\Omega^*(M) \otimes W(\mathfrak{g})$. It follows finally that $\mathfrak{L}(X_\alpha) = i_\alpha D + D i_\alpha$ also vanishes on $\omega^{\#}$ thereby establishing the second condition of a basic class. Thus our assertion is proved in one direction.

Next assume that

$$\omega^{\#} = \omega + \sum \xi_\alpha \theta^\alpha + \frac{1}{2} \sum g_{\alpha\beta} \theta^\alpha \wedge \theta^\beta - \sum g_\alpha u^\alpha \tag{6.15}$$

is a basic closed extension of $\omega^{\#}$. We then have to show that the g_α are the components of a *Poisson lifting*.

From the first basic condition:

$$i_\beta \omega = \xi_\beta \text{ and } i_\beta \xi_\alpha = 0, \quad g_{\beta\alpha} = 0. \tag{6.16}$$

From the assumption $D\omega^{\#} = 0$ it follows that

$$\mathrm{d}g_\alpha = \xi_\alpha \text{ and that } g_{\alpha\beta} = \sum c^\gamma_{\alpha\beta} g_\gamma \tag{6.17}$$

the first by considering terms involving only u_α and the second by considering the terms involving $u_\alpha \theta^\beta$.

This computation then proves the inverse and we have thus established the proposition:

PROPOSITION (6.18). *There is a natural one-to-one correspondence between Poisson liftings of a symplectic action, and equivariant closed extensions $\omega^{\#}$ of the symplectic form ω to $\Omega^*_{\mathfrak{g}}(M)$.*

§7. RELATIONS WITH THE DUISTERMAAT–HECKMAN FORMULA

In the previous section we have reviewed eseentially-known material in a manner which is hopefully accessible to the non-expert in cohomological matters. We come now to an immediate application of these ideas to the push-forward of the symplectic volume under the moment map.

This push-forward $\Phi_*(\omega^n/n!)$ has many beautiful properties, the most notable being that

For any symplectic torus action T which admits a moment mapping

$$\Phi: M \to \mathfrak{g}^* = \mathbb{R}^l, \tag{7.1}$$

the support of $\Phi_*(\omega^n/n!)$ (i.e. the image set $\Phi(M)$) is a convex polyhedron.

This result generalizes inequalities of Schur and Horn relating the eigenvalues and diagonal elements of a Hermitian matrix (and the extension of Kostant to other compact Lie groups), and was established independently by Atiyah on the one hand and Guillemin–Sternberg on the other. (See Guillemin–Sternberg[14] and Atiyah[1].)

As explained in the Introduction the Duistermaat–Heckman result focuses on another very special feature of this measure on $\mathbb{R}^l$, *namely, that it is piecewise polynomial.*

To start off, consider again an action of the circle S on the symplectic manifold (M, ω). Let X be the infinitesimal generator of the action, and assume that the action has a moment map, i.e. there exists a function $f = f_X$ on M such that

$$i(X)\omega = \mathrm{d}f. \tag{7.2}$$

We saw in the last section that once such an f is chosen, then

$$\omega^{\#} = \omega - fu$$

extends ω to a closed equivariant class in $\Omega_S^*(M)$. We may therefore consider the class

$$\mathrm{e}^{\omega^{\#}} = \mathrm{e}^{\omega}\,\mathrm{e}^{-fu} \tag{7.3}$$

in $\hat{H}_T^*(M)$ (where $\hat{H} = H \otimes_{R[u]} R[[u]]$ is now a module over the ring of formal power series)—and apply our integration formula (3.8) to it. This leads to the relation:

$$\pi_*^M \mathrm{e}^{\omega}\,\mathrm{e}^{-fu} = \sum_P \pi_*^P \left\{ \frac{i_P^* \mathrm{e}^{\omega}\,\mathrm{e}^{-fu}}{E(\nu_P)} \right\} \tag{7.4}$$

But recall now that π_* annihilates all but the forms having the dimension of the manifold on which it is applied. Hence the l.h.s. of (7.4) is given by

$$\text{l.h.s.} = \int_M \frac{\omega^n}{n!}\,\mathrm{e}^{-fu},$$

whereas on the r.h.s. we obtain corresponding expressions depending in complexity on the dimension of the fixed point set.

Let us interpret this r.h.s. in the simplest case when F consists of an isolated and hence finite set of points $\{P\}$. As explained in §3 the Euler class $E(\nu_P)$ is then given (see 3.7) by a nonzero multiple of u^n:

$$E(v_P) = \mathrm{e}_P u^n, \qquad \mathrm{e}_P \in \mathbb{Z}.$$

Hence the r.h.s. is given by the expression

$$\sum_P \frac{\mathrm{e}^{-f(P)u}}{\mathrm{e}_P u^n},$$

so that (7.4) goes over into the identity

$$\int_M \mathrm{e}^{-itf}\{\omega^n/n!\} = \frac{1}{(it)^n} \sum_P \frac{\mathrm{e}^{-itf(P)}}{\mathrm{e}_P} \tag{7.5}$$

of the introduction, provided we replace the "indeterminate" u by the "variable" it.

Remarks. (1) The reinterpretation of u as a parameter is essentially the same process described in §5 in connection with Witten's work.

(2) If one wants to avoid formal power series it is of course equivalent to apply (3.8) to the separate powers of $\omega^{\#}$.

(3) We next explain how the cohomological number e_P, is related to the stationary phase expansion of the integral $\mathrm{e}^{-itf}\{\omega^n/n!\}$ near P.

By hypothesis the circle acts on v_P by nontrivial irreducible representations and correspondingly decomposes v_P into the direct sum of 2-planes

$$v_P = \bigoplus V_k \qquad k = 1, \ldots, n,$$

already discussed in §3.

We now introduce linear coordinates $\{x_k, y_k\}$ on V_k which are orthonormal relative to the invariant metric on M_P and oriented in such a manner that the normalized infinitesimal generator X of the action takes the form:

$$X = (2\pi) \sum m_k \left(x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k} \right) \tag{7.6}$$

with the m_k *positive* integers.

By means of the exponential map we can interpret these x_k and y_k also as coordinates near P in M, and then after a rotation in the x_k and y_k (commuting with X) ω will also take on a diagonal form at P, so that near P:

$$\omega = \sum \lambda_k \,\mathrm{d}x_k \wedge \mathrm{d}y_k + \text{higher order} \tag{7.7}$$

with $\lambda_k \neq 0$. We write $(-1)^P$ for the sign of $\lambda_1 \lambda_2 \ldots \lambda_n$, so that this sign measures the compatibility between the orientations of v_P induced by X_P and by ω^n.

Now in this convention the Euler class of V_k is simply $m_k u$, so that the Euler class of v_P, with its proper orientation is given by:

$$E(v_P) = (-1)^P \left(\prod_1^n m_k \right) u^n$$

whence

$$\mathrm{e}_P = (-1)^P \left\{ \prod_1^n m_k \right\}. \tag{7.8}$$

It follows that, up to second order, the Taylor series of f near P has the form:

$$f = f(P) - 2\pi \sum m_k \lambda_k \left(\frac{x_k^2 + y_k^2}{2}\right) + \cdots. \tag{7.9}$$

Hence the Hessian of f at P relative to the x_k and y_k has the determinant:

$$\det Hf_P = (2\pi)^{2n} \prod_1^n (m_k \lambda_k)^2. \tag{7.10}$$

Recall now the stationary phase prescription for computing an approximation to the integral

$$\int e^{itf} \omega^n / n! \tag{7.11}$$

for large t. The recipe (see for instance Woodhouse[18], p. 293–294) is as follows. Every nondegenerate critical point P contributes the leading term:

$$\frac{(2\pi)^n}{t^n} |\det (Hf_P)|^{-1/2} e^{(i\pi/4) \operatorname{sign} Hf_P} e^{itf(P)} \prod \lambda_k \tag{7.12}$$

where sign $H_P f =$ number of positive−number of negative eigenvalues of Hf_P.

In our situation and assuming that precisely l of the λ_k at P are negative, (7.12) goes over into:

$$(1/t)^n e^{i\pi/4\{(2n-2l)-2l\}} e^{itf(P)} \left(\prod_1^n m_k\right)^{-1}, \tag{7.13}$$

as follows from (7.10). Finally in view of (7.8) this simplifies to

$$\left(\frac{i}{t}\right)^n \frac{e^{itf(P)}}{e_P} \tag{7.14}$$

so that the leading term of "stationary phase" takes the form:

$$\int e^{itf} \omega^n / n! \sim \sum_P \left(\frac{i}{t}\right)^n \frac{e^{itf(P)}}{e_P}. \tag{7.15}$$

On the other hand changing the sign of t in our integration formula (7.5) gives the *equality*:

$$\int e^{itf} \omega^n / n! = \sum_P \left(\frac{i}{t}\right)^n \frac{e^{itf(P)}}{e_P}. \tag{7.16}$$

In short (7.5) proves that already *the leading term of the stationary phase approximation gives the precise formula.*

Note that the sign which appears in e_P makes very good sense in terms of the Morse theory of our moment map f. Indeed the number $2l$ above is simply the Morse index of the moment map f at P. That fits in with the discussion at the end of this section.

We shall now compare our proof of (7.16) with that of Duistermaat and Heckman.

As mentioned in the introduction they prove that the push-forward measure $f_*(\omega^n/n!)$ is a piecewise polynomial function. The "breaks" arise from the critical points of f and away from these we can form the symplectic (or Marsden–Weinstein) quotients M_s of M. The push-forward measure, as a function of s, is then just the total symplectic volume of M_s, namely $\int_{M_s} \omega_s^{n-1}/(n-1)!$. The key lemma of Duistermaat–Heckman is then that ω_s is (between breaks) linear in s (as a cohomology class in M_s, which is locally independent of s). In fact ω_s is essentially our ω^* of (6.8)

$$\omega^* = \omega - fu$$

with $f = s$ and u now interpreted as the first Chern class of the S-fibering. Thus formula (6.8) is the key step in both proofs. The proofs only differ in the way in which it is applied, one proof using analytical arguments and the other using more homology theory.

We conclude this section by pointing out a further property of equivariant cohomology in the symplectic case, namely that $H_T^*(M)$ is a *free* module and in particular that it has no *torsion*. This follows from Morse theory applied to a generic component ϕ of the moment map, by arguments explained in (Atiyah–Bott[2]) but which essentially go back to a paper of Frankel[13]. The generalized Morse cells of ϕ give a stratification of M (invariant under T), and $H_T^*(M)$ can be computed from the exact sequences of this stratification because (as one proves) these exact sequences all split into short exact sequences. Since all these exact sequences are built up starting from the cells (which retract onto components of the stationary set F of T), it follows inductively that all terms in the exact sequences (and in particular $H_T^*(M)$) are free modules. The argument also shows that a graded module basis of $H_T^*(M)$ is given by taking a basis for $H^*(F) = \oplus H^*(P)$ and assigning the degree $q + \sigma(P)$ to each $\psi \in H^q(P)$, where $\sigma(P)$ is the Morse index of P.

One can also give a slightly different argument by computing ranks and then showing that the spectral sequence of the fibration $M_T \to BT$ must be trivial. The homological triviality of this fibration is in any case equivalent to the freeness of $H_T^*(M)$. Thus the restriction $H_T^*(M) \to H^*(M)$ is *surjective*. In the de Rham model of §4 this means that every closed T-invariant form ψ on M extends to an equivariant closed form ψ^*. Thus, for the case of a circle action

$$\psi^* = \psi_0 + \psi_1 u + \psi_2 u^2 + \cdots$$

where $\psi_0 = \psi$, each ψ_k is T-invariant and

$$d\psi_k + i(X)\psi_{k-1} = 0 \qquad k \geq 1.$$

These equations express, on the form level, the triviality of the spectral sequence of the fibration $M_T \to BT$. Note in particular, taking $k = 1$, that $i(X)\psi$ is an *exact* form. This is something one might expect to prove purely analytically.

Since $H_T^*(M)$ is free the localization theorem (3.5) is now considerably strengthened, and in particular

$$i^*: H_T^*(M) \to H_T^*(F)$$

is injective.

In Witten's set-up described in §5 the dimension of the null-space W_s of the Hamiltonian H_s is now independent of s, even for $s = 0$.

We have seen in this section how the Duistermaat–Heckman integration formula on symplectic manifolds can be viewed as a special case of more general results about equivariant cohomology. One incidental advantage of this more general viewpoint is that it includes the case of *degenerate* symplectic manifolds, i.e. manifolds where the closed 2-form ω is allowed to degenerate so that ω^n is not required to be everywhere non-zero. The moment map is again defined by (7.2), and the fixed-point contribution is just the same as before. The only point that requires special treatment is the *orientability* of M. If ω^n is nowhere zero so that M is symplectic then M is automatically orientable. However if $\omega^n = 0$ on a subset $S \subset M$ then S represents the first Stiefel–Whitney class of M and M is only orientable if this class in $H^1(M, Z_2)$ is zero. We must therefore make the orientability of M a separate assumption (otherwise there is no fundamental cycle for our integration).

The degenerate case can be applied to obtain integral formulae of the following type. Assume that M is simply-connected, of dimension $2n$, and admits a circle action with isolated fixed points $\{P\}$. Let $x_1, \ldots, x_k$ be a basis of $H^2(M, R)$ and represent these by closed 2-forms $\omega_1, \ldots, \omega_k$. By averaging over the circle action we can assume that all the ω_i are invariant. Since M is simply-connected we can define functions f_i as in (7.2), so that

$$i_x(\omega_i) = \mathrm{d}f_i.$$

They are uniquely determined modulo constants. Now apply the basic integration formula (7.5) with

$$\omega = \sum_{i=1}^{k} \lambda_i \omega_i$$

and equate, for example, the terms independent of t. This gives

$$\int_M \frac{(\sum \lambda_i \omega_i)^n}{n!} = (-1)^n \sum_{\{P\}} \frac{(\sum \lambda_i f_i(P))^n}{n!\, e_P}.$$

Equating the various monomials in the λ_i we deduce that, for any monomial (and hence polynomial) ϕ of degree n in k variables,

$$\int_M \phi(\omega_1, \ldots, \omega_k) = (-1)^n \sum_{\{P\}} \frac{\phi(f_1(P), \ldots, f_k(P))}{e_P}. \tag{7.17}$$

Note that the integral is just the value on the fundamental cycle of the cohomology class $\phi(x_1, \ldots, x_k)$. Thus (7.17) evaluates such "cohomology numbers" in terms of the values of the corresponding functions f_i at the fixed-points. We shall discuss results of this type in more detail in the next section.

§8. ON EQUIVARIANT CHERN NUMBERS

We close with some comments on the relation between the residue formulae of (Bott [7]) which express the Chern numbers of a bundle E, on which a Killing field X acts, in terms of the behavior of X at its fixed point set. All these results come into better focus in the context of the equivariant theory, as we will now explain. For simplicity we restrict ourselves to a circle action (with generator X) and take E to be a complex line bundle on which the circle S is assumed to act compatibly with its action on M. Thus if $\Gamma(E)$ denotes the space of smooth sections of E, and s is such a section of E, then the action of X on

s, written Xs, is a well-defined new section, and the operator X has the derivation property:

$$X(fs) = (Xf)s + fXs, \tag{8.1}$$

relative to multiplication by a function f on M.

Let us take the point of view that a connection on E is a first-order operator

$$D:\Gamma(E) \to \Gamma(E \otimes T^*)$$

subject to the derivation property

$$D(fs) = (\mathrm{d}f)s + fDs; \tag{8.2}$$

and call a connection D *equivariant if the actions of* D *and* X *commute.*

In terms of a generating section of E over U, the action of X on s is described by a function $L(s)$, satisfying

$$Xs = L(s)s, \tag{8.3}$$

while the connection D is described by a 1-form $\theta(s)$ satisfying

$$Ds = \theta(s)s, \tag{8.4}$$

and in terms of these data the equivariance of D is expressed by the identity:

$$\mathrm{d}L + L\theta = \mathfrak{L}(X)\theta + \theta L;$$

that is, by the relation

$$\mathrm{d}L(s) = \mathfrak{L}(X)\theta(s). \tag{8.5}$$

Observe next that equivariant connections always exist for a compact group action. Indeed, if D is an arbitrary connection on E, its transform under an element $g \in G$ of the action is a well-defined new connection on E given by

$$D^g s = g^{-1}D(gs), \tag{8.6}$$

and as connections can be averaged the integral $\int_G D^g\,\mathrm{d}g$ over the compact group G will commute with the action and hence also with X.

With this understood, recall now that, in terms of the connection form $\theta(s)$, the ordinary first Chern class of E is represented by

$$\frac{-\mathrm{d}\theta(s)}{2\pi i} \tag{8.7}$$

on U.

More precisely, this form is seen to be independent of the generating section s chosen, and hence is the restriction to U of a global 2-form $c_1(E; D)$. In the equivariant theory we claim this same property for the expression

$$c_1(E, D; s) = -\frac{1}{2\pi i}\{\mathrm{d}\theta(s) - [L(s) - i(X)\theta(s)]u\} \tag{8.8}$$

in $\Omega_X[u]$. Indeed, let s' be another generating section for E over U. Then we have

$$s' = gs$$

where g is some smooth non-vanishing function, and the derivation properties (8.3) and (8.4) imply that

$$L(s') = i(X)\,\mathrm{d}\ln g + L(s')$$

and

$$\theta(s') = \mathrm{d}(\ln g) + \theta(s),$$

whence $c_1(E, D; s) = c_1(E, D; s')$ on U. Thus (8.8) defines a global class in $\Omega^*[u]$. To check that this expression lies in $\Omega_X^*[u]$ it therefore remains only to verify that

$$\mathfrak{L}(X)\,\mathrm{d}\theta = 0 \text{ and } \mathfrak{L}(X)(L + i(X)\theta) = 0.$$

Both these properties follow readily from the invariance property (8.5).

Finally we see that our form $c_1(E, d; s)$ is closed:

$$\mathrm{d}_X c_1 = 0, \tag{8.9}$$

again as an immediate consequence of (8.5). In short then any *equivariant connection D, defines an equivariant first Chern class*

$$c_1{}^X(E; D) = \omega + fu, \tag{8.10}$$

with ω representing the ordinary first Chern class in Ω^* and f a global function given locally in terms of a generating section s by

$$f = \frac{1}{2\pi i}\{L(s) - i(X)\theta(s)\}. \tag{8.11}$$

We are now on very familiar ground and in particular may apply our integration formula to $c_1{}^X(E, D)$ to obtain the identity:

$$\int_M \mathrm{e}^{\omega}\,\mathrm{e}^{fu} = \sum_P \frac{\mathrm{e}^{f(P)u}}{e_P}, \tag{8.12}$$

assuming isolated fixed points $\{P\}$. This implies the desired formula for the Chern number of E:

$$\int_M c_1(E)^n = \sum_P \frac{f(P)^n}{\mathrm{e}_P}, \tag{8.13}$$

when $\dim M = 2n$, and this formula is now easily identified with the corresponding formula in the paper cited above.

Remarks. (1) When X acts on a vector bundle E and D is an equivariant connection for E, an equivariant curvature K_X is given by the expression

$$K_X = K + Lu \tag{8.14}$$

where K is the usual curvature of E and L is the global endomorphism $L_X: \Gamma(E) \to \Gamma(E)$ given by

$$L_X s = Xs - i(X)Ds. \tag{8.15}$$

This L_X thus extends the intrinsic action of X on the fibers of E over the fixed points of X to all of E. Applying a symmetric polynomial φ to K_X is then seen to produce an equivariant form in Ω_X^*, and the integration formula then evaluates $\int_M \phi(K)$ in terms of $\sum_P(\phi(L)/e_P)$.

Formulae of this type go back to (Bott[7]), but were there, and in the subsequent papers, derived by a different principle. Namely, it is argued there that on the *complement* of the fixed point set $\{P\}$ of the action, one can always find a connection D whose corresponding endomorphism L_X—as defined by (8.15)—vanishes identically. From this observation one then derives an *explicit* local way of writing $\phi(K_X)$ as a boundary on $M - \{P\}$:

$$\phi(K_X) = d\psi(K_X, L_X).$$

The form ψ blows up at the fixed points in a manner involving only local information and so finally the integral $\int_M \varphi(K_X)$ is evaluated by a limiting procedure as $\lim_{\epsilon \to 0} - \int_S \psi$ of integrals over ϵ-small spheres about the fixed points.

This procedure is much more cumbersome than the one we outlined above but has the advantage of being applicable in noncompact situations, where the averaging process is not available. In particular this is the case for *holomorphic* vector fields, which actually were the primary motivation for these residue formulae.

(2) Returning to our original framework of a Poisson action of S on (M, ω), note that if ω has integral periods, then the results of §7 are quite equivalent to those of §8. Indeed, under our assumption, there exists a complex line bundle E with connection D such that

(1) the Chern class $c_1(E, D)$ equals ω,

(2) the action of S lifts to E in such a manner that D is equivariant.

The construction of E and this lifting is of course at the heart of the prequantization procedure of Segal, Kostant and Souriau, see, for instance, Woodhouse[18].

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