4. (a) Prove that if \( n \) is odd, \( n^2 - 1 \) is divisible by 8.

If \( n \) is odd, \( n = 2k+1 \) for some integer \( k \). Then \( n^2 - 1 = 4k(k+1) \). But since \( k \) and \( k+1 \) are consecutive integers, one of them has to be even, thus \( k(k+1) \) is even, say it is \( 2s \) for some integer \( s \). Thus \( n^2 - 1 = 4 \cdot 2s = 8s \). So \( 8|n^2 - 1 \).

(b) Prove that \( 4 \nmid n^2 + 2 \) for any integer \( n \).

\( n \) is congruent to either 0, 1, 2 or 3 (mod 4). So \( n^2 \) is congruent to either 0, 1, 2 \( \equiv 0 \) or 3 \( \equiv 1 \) (mod 4). Hence \( n^2 + 2 \) is congruent to 2 or 3 (mod 4). But \( 4|n^2 + 2 \) iff \( n^2 + 2 \equiv 0 \) (mod 4). This completes the proof.

2. Using induction show that \( \sum_{i=1}^{k} 3i^2 - 3i + 1 = k^3 \).

Argue by induction on \( k \): The statement is true for \( k = 1 \), since \( 3 \cdot 1^2 - 3 \cdot 1 + 1 = 1^3 \). Now assume that the statement holds for \( k \). We will show that it holds for \( k + 1 \), that is \( \sum_{i=1}^{k+1} 3i^2 - 3i + 1 = (k+1)^3 \). Indeed:

\[
\sum_{i=1}^{k+1} 3i^2 - 3i + 1 = \{3(k+1)^2 - 3(k+1) + 1\} + \sum_{i=1}^{k} 3i^2 - 3i + 1 = 3k^2 + 3k + 1 + k^3 = (k+1)^3,
\]

as required.

3. Find a parametrization of the rational points on the hyperbola \( x^2 - 2y^2 = 1 \), starting from the point \( (3, 2) \).

If \( (a, b) \) is any other rational point on the curve, then the line \( y = m(x - 3) + 2 \) with \( m = \frac{2 - a}{3 - b} \) will pass through both \( (3, 2) \) and \( (a, b) \). (Note that this excludes the exceptional case when \( a = 3 \) because \( m = \infty \) is then not defined- but if \( a = 3 \), then \( b \) has to be \( \pm 2 \), that is, we only omit the single point \( (3, -2) \). This is OK, because ALL the other rational points on the curve are obtained by considering lines \( y = m(x - 3) + 2 \) through \( (3, 2) \) and intersecting them with our curve.) Observe in passing that \( (a, b) \) on the curve (except \( (3, 2) \)) gives

\[
\begin{align*}
-4 & = 2m(3m^2 + 6m + 3) - (2m^2 + 6m + 3) - 2m(x - 3), \\
& = 2m(3m^2 + 6m + 3) - (2m^2 + 6m + 3) - 2m(x - 3), \\
& = 2m(3m^2 + 6m + 3) - (2m^2 + 6m + 3) - 2m(x - 3), \\
& = 2m(3m^2 + 6m + 3) - (2m^2 + 6m + 3) - 2m(x - 3),
\end{align*}
\]

The last expression has the advantage that putting \( t = 0 \) which corresponds to \( m = \infty \), gives us back the missing hidden point \((3, -2)\).

4. Prove that any positive integer of the form \( 3k + 2 \) has a prime factor of the same form.

We know that every positive integer can be written as a factor of prime numbers. Given a positive integer \( n = 3k + 2 \) for some \( k \). The prime factors of \( n \) are either of the form \( 3m \), \( 3m + 1 \) or \( 3m + 2 \). Now assume, for a contradiction, that none of those primes are of the form \( 3m + 2 \). If there is a prime factor of the form \( 3m \) - btw, it has to be 3- then this means \( n = 3k + 2 \) is divisible by 3, which is impossible (3|3k but 3 /|2). Thus, all factors have to be of the form \( 3m + 1 \). But any such two numbers is of the same form, hence (by induction) the whole product, namely \( n = 3k + 2 \) is of the form \( 3m + 1 \), again a contradiction. This completes the proof.