Quiz #2

1) Calculate the principal part of the Laurent series near $z=0$ for the function $f(z)=\dfrac{\sin 2z}{z^4}$.

Find $\operatorname*{Res}_{z=0} f(z)$.

2) Calculate $\displaystyle\oint_{|z+\frac{3\pi i}{2}|=\pi} \frac{z^2}{e^z-1}\,dz$.

3) Find a Möbius transformation that maps the disk $\mathcal{D}=\{|z+2|<1\}$ to the half-plane $H=\{\operatorname{Im} z<0\}$.

Quiz 2: Solutions

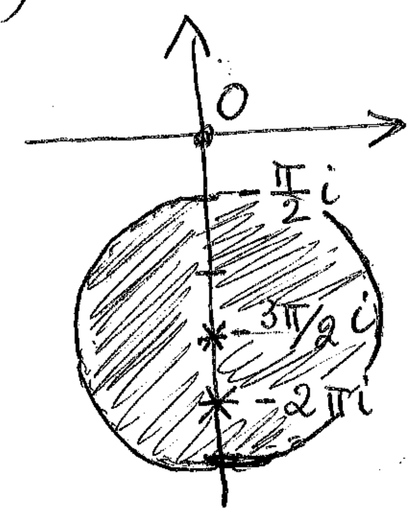
1) $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\sin 2z = 2z - \frac{8z^3}{3!} + \frac{32z^5}{5!} - \dots$$

$$\frac{\sin 2z}{z^4} = \underbrace{\frac{2}{z^3} - \frac{4}{3z}}_{\text{principal part}} + \frac{32z}{5!} - \dots$$

$$\operatorname*{Res}_{z=0} \frac{\sin 2z}{z^4} = -\frac{4}{3}.$$

2) The poles of $\frac{z^2}{e^z-1}$ are <u>zeros</u> of $e^z - 1$, i.e. $z_n = 2\pi i n$, $n = 0, \pm 1, \pm \dots$



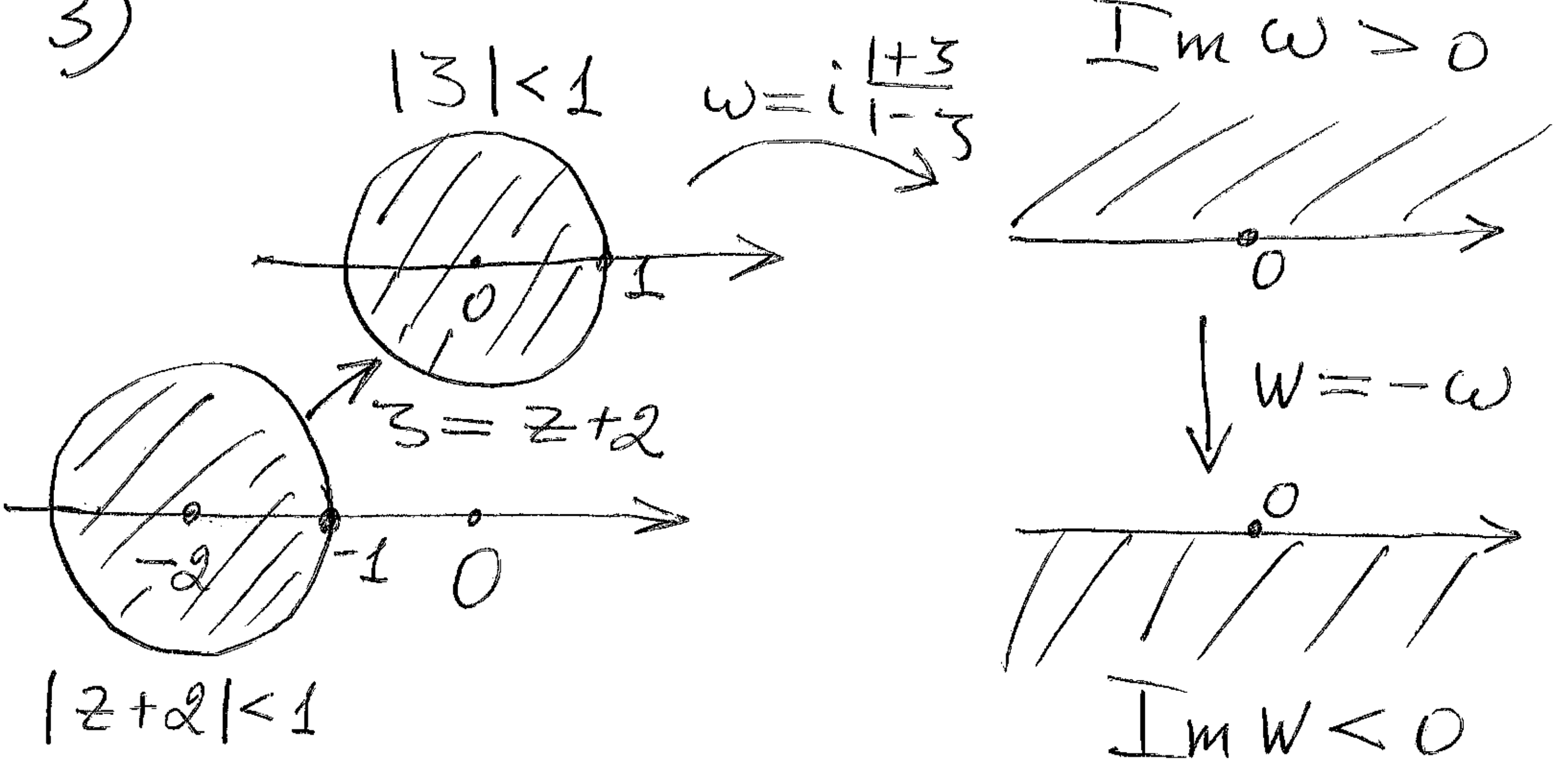
Only one of them, $a = z_{-1} = -2\pi i$ belongs to $\mathcal{D}$.

By the Residue Theorem,

$$\oint \frac{z^2}{e^z-1}\,dz = 2\pi i \operatorname*{Res}_{z=-2\pi i} \frac{z^2}{e^z-1} = 2\pi i \left.\frac{z^2}{e^z}\right|_{z=-2\pi i} = -8\pi^3 i$$

<u>Recall:</u> $\operatorname*{Res}_{z=a} \frac{\varphi(z)}{\psi(z)} = \frac{\varphi(a)}{\psi'(a)}$ as long as $\psi'(a) \neq 0$.

3)



Composing these 3 maps, we obtain:

$$w = -\omega = i\,\frac{\zeta+1}{\zeta-1} = i\,\frac{z+3}{z+1}$$