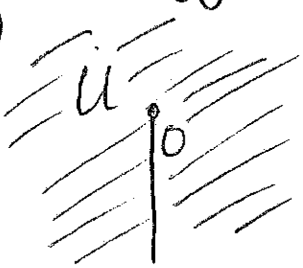
1. a) Write down the expression for the branch of $\log z$ in the slit plane $\mathcal{U} = \mathbb{C} \setminus \{yi : y \le 0\}$ s.t. $\log 1 = 0$. Calculate $\log(-1-i)$ for this branch.




b) Calculate $\displaystyle\int_{\mathcal{C}} \frac{dz}{z}$, where $\mathcal{C}$ is a curve in $\mathcal{U}$ beginning at $1$ and ending at $-1-i$.

2. Expand $\dfrac{1}{(z+1)(z-3)}$ into a power series centered at $0$. Where does it converge?

3. Show that the function

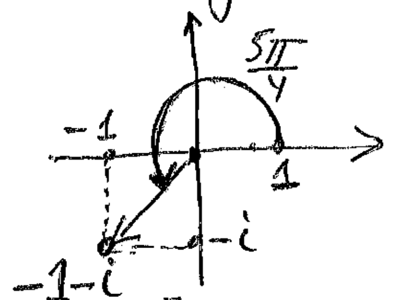
$$f(z) = \begin{cases} \dfrac{e^{-z} - 1 + z}{z^2}, & z \neq 0 \\ \dfrac{1}{2}, & z = 0 \end{cases}$$

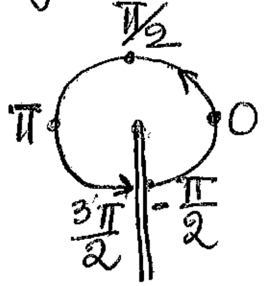
is entire.

4. Calculate $\displaystyle\oint_{|\zeta| = 2\pi} \frac{e^{i\zeta}}{\zeta + \pi}\, d\zeta$.

# Quiz solutions

1. a) $\log z = \log|z| + i\arg z$, where



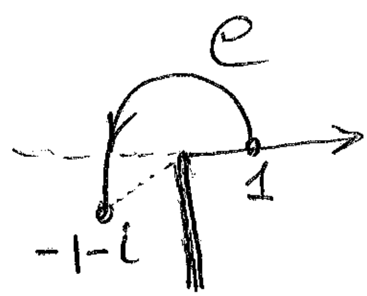

$-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

(so $\arg 1 = 0$)

$-1-i = \sqrt{2}\, e^{i\frac{5\pi}{4}}$

$$\log(-1-i) = \log\sqrt{2} + i\frac{5\pi}{4}.$$

b) $$\int_C \frac{dz}{z} = \log z \Big|_1^{-1-i} = \log(-1-i) - \log 1 = $$
$$= \log\sqrt{2} + i\frac{5\pi}{4}$$




2. $$\frac{1}{(z+1)(z-3)} = -\frac{1}{4}\left(\frac{1}{1+z} + \frac{1}{3-z}\right) =$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}},$$

Taylor expansion

which converges for $|z| < 1$.

[One can also write Laurent expansions in the annulus $\{1 < |z| < 3\}$ and in $\{|z| > 3\}$].

3. $e^{-z} = 1 - z + \frac{z^2}{2} - \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}$

$$e^{-z} - 1 + z = \frac{z^2}{2} - \frac{z^3}{3!} + \frac{z^4}{4!} - \ldots = \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n!}$$

$$\frac{e^{-z} - 1 + z}{z^2} = \frac{1}{2} - \frac{z}{3!} + \frac{z^2}{4!} - \ldots = \sum_{m=0}^{\infty} (-1)^m \frac{z^m}{(m+2)!}$$

This formula is valid for $z \neq 0$.
But the series on the right converges on the whole complex plane, so it gives us an entire function $g(z)$. Moreover, $g(0) = \frac{1}{2} = f(0)$, so $g(z) \equiv f(z)$ everywhere.

4. Since $f(z) = e^{iz}$ is an entire function and the circle $\{|z| = 2\pi\}$ goes around the point $-\pi$, the Cauchy Formula is applicable:

$$\oint_{|z|=2\pi} \frac{e^{iz}}{z+\pi}\, dz = 2\pi i \cdot e^{i(-\pi)} = -2\pi i$$