THE MAXIMUM-ENTROPY MEASURE OF A RATIONAL ENDMORPHISM OF THE RIEMANN SPHERE

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UDC 517.53

Let \( f(z) \) be a rational function of a complex variable regarded as an analytic endomorphism of the Riemann sphere \( S^2 \). In the present note an existence and uniqueness theorem is established for the maximum-entropy measure \( \mu \) of the endomorphism \( f \). We prove that the roots of the equation \( \frac{f^n z}{\mu(z)} = \psi(z) \) are asymptotically equally distributed with respect to the measure \( \mu \), where \( \psi \) is an arbitrary rational function, apart, possibly, from two exceptional constants. In particular, the full inverse images \( \frac{f^n z}{\mu} \) (where \( c \) is not an exceptional constant), as well as the periodic points of the endomorphism \( f \), are asymptotically equidistributed according to the measure \( \mu \).

We shall construct a maximum-entropy measure by investigating a special operator in the space of continuous functions. Let \( A \) be a bounded operator in the complex Banach space \( \mathcal{B} \). We consider its subspaces: \( \mathcal{B}_u \), the closure of the linear hull of the eigenvectors of the operator \( A \) corresponding to unitary eigenvalues \( (|\lambda| = 1) \); \( \mathcal{B}_a = \{ \varphi \in \mathcal{B} \mid A^n \varphi \to 0 \quad (m \to \infty) \} \).

**Definition.** The operator \( A \) is called almost periodic if the orbit \( \{ A^m \varphi \}_{m=1}^{\infty} \) of any vector \( \varphi \in \mathcal{B} \) is strongly precompact.

**Theorem on the Decomposition of a Unitary Discrete Spectrum (see [1]).** If \( A \) is an almost periodic operator in the Banach space \( \mathcal{B} \), then we have the direct decomposition \( \mathcal{B} = \mathcal{B}_a + \mathcal{B}_u \).

**Corollary.** Let the almost-periodic operator \( A \) have no unitary eigenvalues other than 1, and let the subspace of invariant vectors be one-dimensional \( (= \text{Lin}(h)) \). Then there exists an \( A^* \)-invariant functional \( \mu \in \mathcal{B}^* \), \( \mu(h) = 1 \) such that for any vector \( \varphi \in \mathcal{B} \), \( A^m \varphi - \mu(\varphi) h \to 0 \quad (m \to \infty) \).

We apply this result to the following operator: \( A: C(S^2) \to C(S^2) \) in the space of continuous functions:

\[
A \varphi(z) = \frac{1}{n} \sum_{\zeta \in \mu(z)} \varphi(\zeta)
\]

(\( n = \deg f \), \( \varphi \in C(S^2), z \in S^2 \)), where the roots of the equation \( f \zeta = z \) are counted with their numbers of multiplicity. We shall use the concepts of an "irregular point" and an "exceptional point" as defined, e.g., in [2]. By \( F \) we denote the compact of irregular points. By \( I \) we denote the function identically equal to 1. We put \( \| \varphi \|_K = \sup_{x \in K} |\varphi(x)| \).

**Theorem 1.** There exists an \( A^\# \)-invariant probability measure\(^\dagger\) \( \mu \) on the sphere \( S^2 \) such that

\[
\left\| A^m \varphi - \left( \int \varphi d\mu \right) \right\|_K \to 0 \quad (m \to \infty)
\]

\(^\dagger\)By "measure" we shall always mean a complex Borel regular measure.

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for any compact \( K \subset S^2 \) containing no exceptional points of the function \( f \), and any \( \varphi \in C(S^2) \).

We denote by \( \delta_{\zeta} \) a unit mass concentrated as the point \( \zeta \). Let \( \varphi(t) \) be a rational function. We consider the following measures:

\[
\mu_{m, \varphi}^{(\text{con})} = \frac{1}{n^m} \sum_{t \in m^{-1}(\varphi)} \delta_{\zeta}, \quad \mu_{m, \varphi}^{(\text{nc})} = \frac{1}{\lambda_{m, \varphi}} \sum_{t \in m^{-1}(\varphi)} \delta_{\zeta},
\]

where the roots of the equation \( f^m = \varphi(t) \) are counted, respectively, with and without their degrees of multiplicity, and \( \lambda_{m, \varphi} \) is the number of roots counted, ignoring their degrees of multiplicity.

**Theorem 2.** For all rational functions \( \varphi \), apart, possibly, from two exceptional constants, the measures \( \mu_{m, \varphi}^{(\text{con})} \) and \( \mu_{m, \varphi}^{(\text{nc})} \) converge weakly to a certain probability measure \( \mu \) independent of \( \varphi \).

If \( \varphi \equiv c \), where the constant \( c \) is not exceptional, the convergence \( \mu_{m, \varphi}^{(\text{con})} \rightarrow \mu \) \( (m \rightarrow \infty) \) follows from Theorem 1 with \( K = \{c\} \). This result was obtained in [3] for a polynomial \( f \) by methods of potential theory, which apparently do not work in the general case of a rational function.

**Corollary.** The periodic points of a rational endomorphism are asymptotically equidistributed according to the measure \( \mu \).

**Proposition.** a) The carrier of the measure \( \mu \) is the set \( F \) of irregular points. b) If \( F \neq S^2 \), the measure \( \mu \) and the Lebesgue measure are mutually singular.

**Theorem 3.** a) The measure \( \mu \) is \( f \)-invariant. b) The dynamical system \( (f, \mu) \) is exact. c) The entropy \( h_\mu(f) = \log n \).

It was proved in [4] and [5] that the topological entropy \( h(f) = \ln n \). Thus, \( \mu \) is the maximum-entropy measure of the endomorphism \( f \). We remark that the mere existence of a maximum-entropy measure of a rational endomorphism can be deduced from fairly general considerations. Namely, asymptotically h-expansive endomorphisms were defined in [6], and for these the existence of a maximum-entropy measure was established.

**Theorem 4.** Rational endomorphisms of the sphere are asymptotically h-expansive.

**Remark.** A rational endomorphism (and even its restriction to the set \( F \)) is, generally speaking, not h-expansive (for the definition, see [7]).

**Theorem 5.** A rational endomorphism of the Riemann sphere has a unique maximum-entropy measure.

The following lemmas are used to prove Theorem 5:

**Lemma 1.** Let \( f: X \rightarrow X \) be a continuous endomorphism of a metric compact \( X \), and let \( \mu \) be an \( f \)-invariant ergodic probability measure. Let \( Y \subset X \) and \( \mu(Y) > 0 \). Then \( h_\mu(f) \leq h_f(Y) \), where \( h_f(Y) \) is the topological entropy of \( f \) with respect to \( Y \), as defined in [7].

Let \( E = \{1, \ldots, n\} \), \( 0 \leq k \leq 1 \). We consider the set \( \mathcal{G}_m(\varphi) \subset E^m \) of all sequences of length \( m \) in which the element 1 appears at least \( km \) times.

**Lemma 2.** Let \( \varphi \geq \varphi_n \). Then there exist \( K > 0 \) and \( 0 < \theta < n \) such that \( |G_m(\varphi)| \leq K \theta^m \).

We shall say that a system of sets distinguishes the measures \( \mu \) and \( \nu \) if \( \mu(Z) \neq \nu(Z) \) for some set of the system.

**Lemma 3.** Let the measure \( \nu \) be mutually singular with the measure \( \mu \) constructed in Theorem 1. There exists such a natural \( r \) and such a covering \( D = \{D_i\}_{i=1}^n \) of the sphere by closed sets that \( D_i = \overline{D_i^0} \); 2) \( D_i \cap D_j = \emptyset \) \( (i \neq j) \); 3) \( (\mu + \nu)(\cup \partial D_i) = 0 \); 4) the intersection of \( D_i^0 \) and \( f^{-r}z \) is composed of not more than one point \( (1 \leq i \leq n, z \in S^2) \); 5) the covering \( D \) distinguishes \( \mu \) from \( \nu \).

**Added in Print.** The results in this note were presented at the Fifteenth Voronezh Mathematical Winter School (January, 1981); the theses are deposited with the All-Union Scientific and Technical Information Institute (VINITI) (No. 5691, pp. 65–66).
INVARIANT ORDERINGS IN SIMPLE LIE GROUPS. THE
SOLUTION TO É. B. VINBERG'S PROBLEM

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UDC 519.46

Let $\mathfrak{g}$ be a real simple noncompact Lie algebra; $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ the Cartan decomposition; $G$ and $K$ simply connected Lie groups corresponding to the algebras $\mathfrak{g}$ and $\mathfrak{t}$. We assume that $D = G/K$ is a bounded symmetric domain. It is known [1] that in this case the set $\text{Con}$ of all closed convex $G$-invariant cones in $\mathfrak{g}$ distinct from $\{0\}$ and $\mathfrak{g}$ is not empty. There is a maximal and a minimal cone, $C_{\text{max}}$ and $C_{\text{min}}$ in $\text{Con}$; they are unique up to a multiplication by $-1$. The set $\text{Con}$ has been described in [2] and [3], [4] (also see Sec. 2).

Let $C \in \text{Con}$ and let $P = P(C)$ be the closed semigroup in $G$ topologically generated by the set $\exp C$. It defines an invariant partial ordering in $G$ for which $\{g \in G : g \geq e\}$ coincides with $P$. The ordering is nontrivial if $P \neq G$. É. B. Vinberg [1] proved that $P \neq G$ and $P \cap P^{-1} = \{e\}$ for $C = C_{\text{min}}$ and raised the question whether it was true for all $C \in \text{Con}$. S. Paneitz [3] proved that $P(C) \neq G$ for all $C \in \text{Con}$ if $D$ is a classical domain of tubular type; the definition can be found in [5].

**Theorem.** (i) $P(C) \subseteq G \Leftrightarrow C \subseteq C_0$ (the cone $C_0 \in \text{Con}$ is defined in Sec. 3), with $C_0 = C_{\text{max}}$ for tubular $D$ and $C_0 \neq C_{\text{max}}$ for nontubular $D$. (ii) If $P = P(C) \neq G$, then $P \cap P^{-1} = \{e\}$ and the "tangent cone" $C(P)$ (cf. [1]) coincides with $C$.

If $\mathfrak{g} \neq \mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{su}(2, i)$, $\mathfrak{e}_{6/7}$, then $C_0 \neq C_{\text{min}}$ and the existence of a continuum of invariant orderings in $G$ follows from (ii).

1. **Notation.** $\mathfrak{t} \subseteq \mathfrak{h}$ is the one-dimensional center; $\mathfrak{h} \subseteq \mathfrak{t}$, Cartan subalgebra; $\mathfrak{h}_\mathfrak{Re} = \mathfrak{h}$, Weyl group for $(\mathfrak{g}, \mathfrak{h})$; $\Delta^+ \subseteq \mathfrak{h}_\mathfrak{Re}$, set of noncompact positive roots; $\{\alpha_1, \ldots, \alpha_r\} \subseteq \Delta^+$ family of pairwise orthogonal roots, where $r = \text{rank } D$; $E_k$ and $E_{-k}$, root vectors corresponding to the roots $\pm \alpha_k$ ($k = 1, \ldots, r$) normed in such a way that $H_k = [E_k, E_{-k}] = 0$; $X_k = E_k + E_{-k} = 0$, $i(E_k + E_{-k}) \in \mathfrak{p}$, $\alpha_k(H_k) = 2$; $Z$, element of $\mathfrak{t}$ defined by the condition that $\alpha(Z) = 2 \forall \alpha \in \Delta^+$; $H_0 = Z - H_1 - \ldots - H_r$; and $\langle , \rangle$, Killing form.

2. Information from [1], [2]. Each $C \in \text{Con}$ lies in $\overline{\mathfrak{t} + M}$ and is uniquely determined by the cone $c = iC \cap \mathfrak{h}_\mathfrak{Re}$ (we write $C \leftrightarrow c$); here $c$ contains either $Z$ (then we write $C \subseteq \text{Con}^+$) or $-Z$. If $c^* = \{X \in \mathfrak{g} : \langle X, \mathfrak{h} \rangle > 0 \forall Y \in c\}$, $c^* = \{X \in \mathfrak{h}_\mathfrak{Re} : \langle X, \mathfrak{h} \rangle > 0 \forall Y \in c\}$, then $C \leftrightarrow c$ implies $C^* \leftrightarrow c^*$. Let $c_{\text{min}} \subseteq \mathfrak{h}_\mathfrak{Re}$ be the cone spanned by $\Delta^+$ and $c_{\text{max}} = c_{\text{max}}^*$; then $C_{\text{min}} \leftrightarrow c_{\text{min}}$ and $C_{\text{max}} \leftrightarrow c_{\text{max}}$. If $c \subseteq \mathfrak{h}_\mathfrak{Re}$ is a closed convex cone, then $(c \leftrightarrow C$ for some $C \subseteq \text{Con}) \leftrightarrow (Wc = c, c_{\text{min}} \subseteq c \lor \text{c}_{\text{max}}).$

3. The Definition of the Cone $C_0$. $C_0 \leftrightarrow c_0$, where $c_0^* = c_{\text{max}}^*$ is the cone spanned by $c_{\text{min}}$ and $Wc_0$.

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