where $k_0 = D^{-1}[\pi p + (q + 1/2)\alpha]$, $\alpha_{n+1}(t)$ is a polynomial of degree n + 1, and $\alpha = \arccos\sqrt{1 - 2A_2D}$. We choose A_2 by setting $\alpha = \pi m_1/m_2$, where m_1/m_2 is an irreducible fraction and $m_2 > 2M_0m_1$ (the arithmetical condition). We require that the coefficients of t^{n+1} in the polynomials $\alpha_{n+1}(t)$, $1 \le n \le M_0 - 1$, are equal to 0. Hence, the remaining coefficients in (2) are uniquely determined.

The constructed diameter corresponds to the elliptic periodic trajectory of the billiard that has very strong degeneracy. Let us find out of what type is the corresponding part of the spectrum of the Laplace operator. Let us set $p = (m - s)m_1$, $q = m_2s$. We get the following formula for $\lambda_{ms} = k_{pq}^2$ from (5):

$$\lambda_{ms} = am^2 + bm + d + \frac{es}{m}(1 + O(m^{-\varepsilon})) + O(m^{-1}), \tag{6}$$

where the estimates $O(m^{-\varepsilon})$ and $O(m^{-1})$ are uniform with respect to s such that $0 \le s \le \mathrm{const}\ m^{1-\varepsilon}$,

$$a = \alpha^3(\mathrm{ctg}^2\,\alpha)m_2^2 \ne 0, \quad e = \alpha^{-1}\left(\frac{1}{192}\,\mathrm{ctg}\,2\alpha + \frac{1}{16}\sin 2\alpha + \frac{11}{384}\sin 4\alpha\right) \ne 0.$$

It follows from (3) that for a certain $c_1 > 0$ each interval $[\lambda_{ms} - c_1m^{-2}, \lambda_{ms} + c_1m^{-2}]$ contains at least one eigenvalue of $-\Delta$. Since $0 \le s \le \mathrm{const}\ m^{1-\varepsilon}$, the considered spectrum is divided into narrow groups of dimension $\sim m^{-\varepsilon}$, the group near λ_{m0} contains $N_m \ge \mathrm{const}\ m^{1-\varepsilon}$ eigenvalues, the distance between which $\sim m^{-1}$. If the estimate $r(\lambda) = O(\lambda^{(1/2)-\delta})$, $\varepsilon < 2\delta < 1 + \varepsilon$, is valid, then it would follow from (1) that $N_m = O(m^{1-2\delta})$, which contradicts the above lower estimate.

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ENTROPY OF ANALYTIC ENDOMORPHISMS OF THE RIEMANNIAN SPHERE

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1. Bowen has asked the following question in the survey [1]: Let P(z) be a complex polynomial, considered as a continuous mapping $S^2 \to S^2$. Is it true that the topological entropy h(P) is equal to ln (deg P)? In the present note we obtain an affirmative answer to this question and, in addition, prove the formula h(f) = ln (deg f) for all rational functions f(z) ($\ne$ const). In this connection, deg f is defined as the maximum of the degrees of the numerator and the denominator in the irreducible representation of f(z).

2. Let M be a smooth manifold equipped with the Riemannian metric d, $Z \subset M$, and $B(Z; r) = \{\zeta \mid \zeta \in M, d(\zeta, Z) < r\}$ $(r > 0)$. For arbitrary ε, δ, k $(0 < \varepsilon < \delta, 0 < k < 1)$ and an arbitrary finite $Z = \{z_1, \ldots, z_n\}$ let $\gamma(\delta, k, Z, \varepsilon)$ denote the smallest number of points $u_1, \ldots, u_\gamma \in M$ such that $\bigcup_{1 \le i \le \gamma} B(u_i;\ kd\ (u_i,\ Z)) \supset B(Z;\delta) \setminus B(Z;\varepsilon)$.

LEMMA 1. If $M = \mathbf{R}^2$ and d is the Euclidean metric, then

$$\gamma(\delta, k, Z, \varepsilon) \le c(k)n^2(\ln(\varepsilon^{-1}\delta) + a(k)). \tag{1}$$

Proof. The boundary $\partial B(Z; r)$ is the union of circular arcs that are less than n^2 in number. Marking equidistant points $v_1, \ldots, v_s$ with angular step k on each of these arcs, we get $\bigcup_{1 \le i \le s} B(v_i; kr) \supset B(Z; \alpha r) \setminus B(Z; r)$, where $\alpha = \alpha(k)$ depends only on k (for reasons of symmetry) and $s < c_0(k)n^2$. Using this construction for $r = \alpha^i\varepsilon$ $(i = 0, 1, \ldots, [(\ln\alpha)^{-1}\ln(\varepsilon^{-1}\delta)])$, we get the desired system of points.

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The following proposition follows from Lemma 1.

Proposition 1. If M is a two-dimensional compact Riemannian manifold, then there exists a $\delta_0 = \delta_0(M)$ such that estimate (1) is valid for $\delta < \delta_0$.

Now let $M = S^2$ be the unit sphere in $\mathbf{R}^3$ with the induced metric d. The stereographic projection identifies S^2 with $\overline{\mathbf{C}}$.

Proposition 2. Let $0 < \eta < \pi/2$, $\delta > 0$. Then there exists a $k = k(\eta, \delta)$ $(0 < k < 1)$ with the following property: If $h: B(u; \rho) \to S^2$ $(0 < \rho < \pi)$ is a univalent meromorphic function that does not take values in some η-net in S^2, then $hB(u; k\rho) \subset B(hu; \delta)$.

Proof. Let $v = hu$, w be the diametrically opposite point, and $\xi \in B(w; \eta)$ be a point of the η-net. Let us consider the bilinear transformations φ and ψ that preserve the metric d and are such that $\varphi(\xi) = \infty$ and $\psi(u) = 0$. The function $g = \varphi h \psi^{-1}$ is holomorphic and univalent in the disk $B(0; \rho)$. If $\omega = g(0)$, then $d(\omega, 0) = \pi - d(\omega, \infty) = \pi - d(v, \xi) < \eta$. By virtue of our conditions, the function g has an exceptional value ζ such that $d(\zeta, \omega) < \eta$. Since

$$(2\cos^2\eta)\,|z_1 - z_2| \leqslant d(z_1, z_2) \leqslant 2\,|z_1 - z_2|$$

in the disk $B(0, 2\eta)$ $(0 < \eta < \pi/2)$, it follows that $|\zeta - \omega| \leqslant \eta/2\cos^2\eta$.

Let us denote the Euclidean disk of radius R with center at a by $D(a; R)$. Let $M(R)$ be the Euclidean radius of the disk with center ω that is circumscribed about the domain $gD(0; R)$ and $m(R)$ be the radius of the inscribed disk. By virtue of what we have proved above, $m(\operatorname{tg}\frac{\rho}{2}) \leqslant \eta/2\cos^2\eta$. It follows from the Koebe distortion theorem (for $r = 1/2$) that $M\left(\frac{1}{2}\operatorname{tg}\frac{\rho}{2}\right) \leqslant 9m\left(\frac{1}{2}\operatorname{tg}\frac{\rho}{2}\right) \leqslant 9\eta/2\cos^2\eta$. If $0 < k < \min\left(\frac{1}{2}, \delta\cos^2\eta/18\eta\right)$, then by the Schwartz lemma we get $M\left(k\operatorname{tg}\frac{\rho}{2}\right) < \delta/2$, i.e., $gD\left(0; k\operatorname{tg}\frac{\rho}{2}\right) \subset D\left(\omega; \frac{\delta}{2}\right)$. Hence $gB(0; k\rho) \subset B(\omega; \delta)$. Reverting to h, we get the desired inclusion.

3. THEOREM 1. Let $f: S^2 \to S^2$ be a rational function that is not a constant. Then the topological entropy $h(f) = \ln(\deg f)$.

Proof. The lower estimate $h(f) \geq \ln(\deg f)$ follows from the Misiurewicz–Przytycki theorem [2].

Let us fix $\eta \in (0, \pi/2)$ and consider the finite η-net $X \subset S^2$, that contains all the critical points of the mapping f. We set $Z_m = \bigcup_{1 \leqslant i \leqslant m} f^i X$ $(m = 1, 2, 3 \ldots)$. Let $\delta < \delta_0(S^2)$ (see Proposition 1), $k = k(\eta, \delta)$ (see Proposition 2), and $\varepsilon_m = \delta L^{-m}$, where L is the Lipschitz constant of the mapping f in the spherical metric. By Proposition 1, $\gamma(\delta, k, Z_m, \varepsilon_m) = O(m^3)$. Let $u_1, \ldots, u_\gamma$ be the corresponding points and $u_{\gamma+1}, \ldots, u_\beta$ be a minimal $(k\delta)$-net for $S^2 \setminus B(Z_m; \delta)$. Then $\beta = \gamma + O(1) = O(m^3)$. If $\rho_m(u) = \min(d(u, Z_m), \delta)$, then the disks $B(u_i; k\rho_m(u_i))$ $(1 \leqslant i \leqslant \beta)$ cover $S^2 \setminus B(Z_m; \varepsilon_m)$. On the other hand, since the critical values of the mapping f^m are contained in Z_m, it follows that all the branches f_λ^{-i} $(1 \leq i \leq m,\ 1 \leq \lambda \leq (\deg f)^m)$ exist and are analytic in the disks $B(u_i; \rho_m(u_i))$ $(1 \leq i \leq \beta)$. All the branches f_λ^{-i} are univalent.

Let $z \in S^2$. We define the number j $(-1 \leq j \leq m)$ as the smallest number such that $f^{j+1}z \in B(Z_m; \varepsilon_m)$ [in the case where $f^i z \notin B(Z_m; \varepsilon_m)$ $(i = 0, \ldots, m)$, we set $j = m$]. Let us suppose that $-1 < j < m$. Then there exists a $z_\nu \in Z_m$ such that $f^{j+1}z \in B(z_\nu; \varepsilon_m)$, and a u_μ $(1 \leq \mu \leq \beta)$ such that $f^j z \in B(u_\mu; k\rho_m(u_\mu))$. The "end" $\{f^i z\}_{j+1}^m$ of the trajectory of the point z is δ-compressed by the trajectory $\{f^{i-j-1}z_\nu\}_{j+1}^m$ of the point z_ν, since $\varepsilon_m L^{i-j-1} < \delta$ $(i = j+1, \ldots, m)$. For the compression of the initial part $\{f^i z\}_0^j$ we select the branches $f_{\lambda_i}^{-i}$ $(1 \leq i \leq j)$ in the disk $B(u_\mu; \rho_m(u_\mu))$ by the conditions $f_{\lambda_i}^{-i}\xi = f^{j-i}z$ $(\xi \equiv f^j z)$. The indicated part is δ-compressed by the corresponding part of the trajectory of the point $v_{\mu,\lambda_j} \equiv f_{\lambda_j}^{-j}u_\mu$. Indeed, the functions $f_{\lambda_{j-i}}^{-(j-i)}$ $(0 \leqslant i \leqslant j)$ do not take values in the η-net X. By Proposition 2, $f^i z = f_{\lambda_{j-i}}^{-(j-i)}\xi \in B(f_{\lambda_{j-i}}^{-(j-i)}u_\mu; \delta)$. But $f_{\lambda_{j-i}}^{-(j-i)}u_\mu = f^i v_{\mu,\lambda_j}$.

For $j = -1$ the whole trajectory is δ-compressed by the trajectory of the point z_ν, and for $j = m$ it is δ-compressed by the trajectory of the point v_{μ,λ_j}.

In the sequel we will use the definition of the topological entropy introduced in [3]. Let us consider the covering of U_δ by all δ-disks and form the covering $U_\delta^m \equiv \bigvee_{0 \leqslant i \leqslant m} f^{-i}U_\delta$. It follows from what we have proved that the smallest number of elements in subcoverings of the covering U_δ^m satisfies the inequality $N(U_\delta^m) \leqslant |Z_m| \times \beta \sum_{0\leqslant j\leqslant m} (\deg f)^j = O(m^b (\deg f)^m)$. Hence $h(f, U_\delta) \leq \ln(\deg f)$. Consequently, $h(f) \leq \ln(\deg f)$.

Remark 1. In the case where f is a polynomial, the preparatory part of the proof is somewhat simplified due to the localization of the dynamics in the finite part of the plane **C**.

Remark 2. Let $\mathfrak{H}_l$ be the class of the rational functions of degree l for which the trajectories of all the critical points converge to attracting cycles. In [4-6] a symbolic dynamics for the polynomials of the class $\mathfrak{H}_l$, and in [7] that for all $f \in \mathfrak{H}_l$, has been constructed. The desired estimate $h(f) \leq \ln l$ for $f \in \mathfrak{H}_l$ follows immediately from these results. However, this estimate also follows easily from the fact that the family of all the branches f_λ^{-i} exists and is normal in the neighborhood of each irregular point for $f \in \mathfrak{H}_l$.

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PROPERTIES OF A CLASS OF NON-SELF-ADJOINT OPERATORS

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The present paper is devoted to an extension to the case of abstract operators of a well-known result of Pavlov [1] which states that if the function q: $\mathbf{R}^3 \to \mathbf{C}$ is continuous and if $|q(x)| \exp(\varepsilon |x|^{1/2}) \leqslant c$ for some c, $\varepsilon > 0$, then the operator $-\Delta + q$, regarded as an operator in $L_2(\mathbf{R}^3)$, has a finite number of eigenvalues.

Let $H = L_2(\mathbf{R}; E)$, where E is a separable Hilbert space. Following [2], we consider the operator T: H → H associated with the sesquilinear form*

$$(f, g)_T = (f, S^n g)_H + (BVf, AUg)_G, \quad f \in D(V), \quad g \in D(S^n), \tag{1}$$

where S is the operator of multiplication by the independent variable in the space H, $n \in \mathbf{N}$, $U = (S^2 + I)^{\nu/2}$, $V = (S^2 + I)^{(n-1-\nu)/2}$, $0 \leqslant \nu \leqslant n - 1$, and the operators A and B belong to $\mathscr{B}(H; G)$† (G is some auxiliary Hilbert space). Let F be the Fourier–Plancherel operator in H, and let ε be an arbitrary but fixed number in the interval]0; 1[. We denote by $\mathscr{H}_\delta$, where $0 < \delta \leq 1$, the countable Hilbert space whose elements are those $f \in H$, for which

$$\|f\|_{\delta,\gamma} \overset{\text{def}}{=} \left\{ \int_{-\infty}^{\infty} \exp(2\gamma |x|^\delta) \|Ff(x)\|_E^2 dx \right\}^{1/2} < \infty$$

for all $\gamma \in]0; \varepsilon[$. The space $\mathscr{H}_\delta$, furnished with the topology generated by the norms $\|\cdot\|_{\delta,\gamma}$, $\gamma \in]0; \varepsilon[$, is everywhere dense and continuously imbedded in H.

*D(·) denotes the domain of the operator ·.
†$\mathscr{B}(H; G)$ ($\mathscr{B}_\infty(H; G)$) is the space of all continuous (completely continuous) linear operators from H into G.

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