SPECTRAL THEORY OF ALMOST-PERIODIC REPRESENTATIONS OF SEMIGROUPS

M. Yu. Lyubich and Yu. I. Lyubich

UDC 519.4

1. A representation T of a topological semigroup S in a Banach space $\mathfrak{B}$ is said to be almost periodic (a.p.) if $\forall x\in\mathfrak{B}$ the orbit $O(x)=\{y\mid y=T(s)x\,(s\in S)\}$ is strongly precompact, or equivalently, the strong closure $B_T = \overline{T(S)}$ is strongly compact. A compact semigroup $B_T$ is said to be a Bohr compactum, and its smallest two-sided ideal $K_T$ is called the Sushkevich kernel of T. For these basic definitions, see [1]. Below we assume that $K_T$ is a group. This is valid, in particular, if S is Abelian. The identity $P\in K_T$ is a projector in $\mathfrak{B}$. We call it the boundary projector, and $\mathfrak{B}_1=\operatorname{Im}P$, $\mathfrak{B}_0=\operatorname{Ker}P$ the boundary and inner subspace of T. Obviously, $\mathfrak{B}=\mathfrak{B}_1+\mathfrak{B}_0$, and we can prove that $\mathfrak{B}_1$ and $\mathfrak{B}_0$ are invariants. If T is contractive (we can always arrange this by going over to an equivalent norm in $\mathfrak{B}$), then $\|P\| = 1$ for $P\neq 0$, that is, P is an orthoprojector. In the development of [1, 2] we establish the following theorem on the removal of the boundary spectrum.

THEOREM 1. The inner and boundary subspaces are described as follows:

$$\mathfrak{B}_0=\{x\mid 0\in\overline{O(x)}\},\quad \mathfrak{B}_1=\overline{\sum_\lambda V_\lambda}, \tag{1}$$

where $V_\lambda$ runs through finite-dimensional invariant subspaces for which $T|V_\lambda$ are irreducible and unitary up to equivalence. If T is contractive, then $T|\mathfrak{B}_1$ is an isometric representation, and $B_T|\mathfrak{B}_1=\overline{(T|\mathfrak{B}_1)(S)}$ is a compact (in the strong topology) group of isometries.

The proof is based on the fact that 1) $A\to AP$ is a homomorphism-retraction $B_T\to K_T$ and 2) for Banach representations of compact groups the theory of Peter and Weyl is valid in a suitable form.

Suppose that S is Abelian. Then it can be converted to a directed set by putting $s\geqslant t$ if s is divisible by t or s = t. It turns out that

$$\mathfrak{B}_0=\{x\mid \lim_s T(s)x=0\}. \tag{2}$$

Furthermore, the theorem on the removal of the boundary spectrum with the refinement (2) for $\mathfrak{B}_0$ is valid for the wider class of asymptotic almost periodic (a.a.p.) representations. This class is defined as follows: T is bounded [that is, $\sup_s\|T(s)\|<\infty$ ] and the direction $\{T(s)\}$ is asymptotically strongly precompact (that is, any subdirection of it contains a strongly convergent subdirection).

The Sushkevich kernel $K_T$ for an a.a.p. T is constructed as the strong $\omega$-limit set of the direction $\{T(s)\}$. The Sushkevich kernel is a compact Abelian group. The identity P of this group is called (as before) the boundary projector. It is interesting to note that the removal of the boundary spectrum [with the refinement (2)] implies asymptotic almost periodicity, so that in a finite calculation these properties are equivalent. We also note that because S is Abelian

$$\mathfrak{B}_1=\overline{\sum_\chi{}^{\cdot} W_\chi}, \tag{3}$$

where $W_\chi=\{x\mid T(s)x=\chi(s)x\,(s\in S)\}$ are the weighted subspaces corresponding to the unitary weights $\chi$ (that is, unitary characters of the semigroup for which $W_\chi\neq 0$).

A bounded representation T is said to be convergent if the direction $\{T(s)\}$ is strongly convergent.

THEOREM 2. For the representation T to be convergent it is necessary and sufficient that it should be a.a.p. and should not have unitary weights other than 1. Also,

Kharkov State University. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 36, No. 5, pp. 632-636, September-October, 1984. Original article submitted April 18, 1983.

 0041-5995/85/3605-0474$08.50 

$\lim_{s} T(s) = P_1$, where $P_1$ is the projector onto the subspace of fixed vectors $W_1 = \{x \mid T(s)x = x(s \in S)\}$, $\operatorname{Ker} P_1 = \mathfrak{B}_0$.

As an example we consider the representation $n \mapsto A^n$, $A \in \operatorname{End} \mathfrak{B}$, of the additive semigroup $\mathbb{Z}_+$. If it is a.p., then A is called an a.p. operator (in this case, a.a.p. is no different from a.p.). An example is a compact operator with bounded powers. A more general situation is associated with a theorem of Yosida and Kakutani [3]: $\sup_{n \geq 1} \|A^n\| < \infty$, $r(A) = 1$ [$r(\cdot)$ is the spectral radius], $\exists m$: $A^m = C + R$, where C is compact and $r(R) < 1$. For such operators we can verify directly that they are a.p., which gives a new proof of the theorem of Yosida and Kakutani.

If A is an algebraic operator with spectrum in the unit disk and all the roots of the minimal polynomial that lie on the unit circle are simple, then A is an a.p. operator. In particular, any projector is an a.p. operator.

Let us state the results that follow from Theorems 1 and 2 that apply to operators.

COROLLARY 1. If A is an a.p. operator in a Banach space $\mathfrak{B}$, then $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_0$, where $\mathfrak{B}_1$ is the closure of the linear hull of the eigenvectors that correspond to unitary eigenvalues, and $\mathfrak{B}_0 = \{x \mid \lim_{n\to\infty} A^n x = 0\}$. If $\|A\| \leq 1$, then the projector P that effects this decomposition ($\operatorname{Im} P = \mathfrak{B}_1$, $\operatorname{Ker} P = \mathfrak{B}_0$) is orthogonal [$P = 0$ for $\|A\| < 1$ and even for $r(A) < 1$], and the operator $A|\mathfrak{B}_1$ is an invertible isometry.

COROLLARY 2 (cf. [4]). For the strong $\lim_{n\to\infty} A^n$ to exist it is necessary and sufficient that A should be a.p. and should not have unitary eigenvalues other than 1. Also, $\lim_{n\to\infty} A^n = P_1$, where $P_1$ is the projector onto the subspace of fixed vectors of the operator A such that $\operatorname{Ker} P_1 = \overline{\operatorname{Im}(I - A)}$.

This result can be regarded as a strengthening of the statistical ergodic theorem in the a.p. situation. Because of the almost periodicity, no restrictions on the Banach space $\mathfrak{B}$ arise here.

Similarly we can consider an a.p. representation $t \to U_t$ of the additive semigroup $\mathbb{R}_+$ (that is, one-parameter a.p. semigroups in $\mathfrak{B}$). Omitting the statements of the results, we merely note that from the theorem on the removal of the boundary spectrum there follows the well-known theorem that on the semiaxis $t \geq 0$ any a.p. function (a.p.f.) $\varphi(t)$ has the form $\varphi = \varphi_1 + \varphi_0$, where $\varphi_1$ is the uniform limit of linear combinations of exponents $e^{i\lambda t}$, $\lambda \in \mathbb{R}$ ($\varphi_1$ is thereby extended to an a.p.f. on the whole axis), and $\varphi_0$ tends to zero as $t \to +\infty$. This is the general form of an a.p.f. on the semiaxis, since the converse assertion is trivial.

2. In a Banach space $\mathfrak{B}$ with cone $\mathfrak{C}$ we can naturally distinguish the class of nonnegative representations, that is, those such that $T(s) \geq 0$ for all $s \in S$. The following generalization of the classical theorem of Perron and Frobenius is true for this class.

THEOREM 3. Let $\mathfrak{B}$ be a Banach space with total cone $\mathfrak{C}$ (that is, $\mathfrak{B} = \overline{\mathfrak{C} - \mathfrak{C}}$). Let T be a nonnegative a.p. representation of a topological semigroup S for which the Sushkevich kernel $K_T$ is a group. We assume that there is a vector v whose orbit is nonzero, that is, $\inf_S \|T(s)v\| > 0$. Then there are an invariant vector $h \geq 0$ and an invariant linear functional $\mu \geq 0$ such that $\mu(h) = 1$.

We shall call h, $\mu$ a Perron–Frobenius pair (PF-pair) for T. Let us describe the construction of a PF-pair under the conditions of Theorem 3. We observe that $P \neq 0$, since $Pv \neq 0$. But then there is an $x_0 \geq 0$ such that $Px_0 \neq 0$. Let us put $h = \int_{K_T} (Ax_0)\, dA$, $\mu_0 = \int_{K_T} \lambda(A\,\cdot)\, dA$, where dA is the normalized Haar measure on $K_T$, $\lambda \in \mathfrak{C}^*$, $\lambda \geq 0$, $(\lambda P x_0) > 0$. The elements h and $\mu_0$ are invariant, and $\mu_0(h) = \lambda(h) = \int_{K_T} \lambda(Ax_0)\, dA > 0$. It remains to put $\mu = \mu_0 / \mu_0(h)$.

Remark. If S is Abelian, it is sufficient that T should be a.a.p.

COROLLARY 3. Let $\mathfrak{B}$ be a Banach space with a total cone, and $A \geq 0$ an a.p. operator. Then either $A^n$ tends strongly to zero as $n \to \infty$, or A has a PF-pair h, $\mu$: $Ah = h \geq 0$, $A^*\mu = \mu \geq 0$, $h(\mu) = 1$.

This assertion for compact $A \geqslant 0$ gives the well-known Krein–Rutman theorem (with the condition of boundedness of powers). A more general theorem about the existence of a PF-pair has been obtained in the Yosida–Kakutani situation.

If the cone $\mathfrak{C}$ is solid (that is, $\mathfrak{B}$ is a Krein space), then the representation $T \geqslant 0$ is said to be indecomposable if $\forall x \geqslant 0\ (x \neq 0)\ \forall \omega \in \mathfrak{B}^*,\ \omega \geqslant 0\ (\omega \neq 0)\ \exists s \in S:\ \omega(T(s)x) > 0$, and primitive if $\forall x \geqslant 0\ (x \neq 0)\ \exists s \in S:\ T(s)x > 0$ [that is, T(s)x is an interior point of the cone]. These concepts carry over to operators $A \geqslant 0$ through the corresponding representations.

LEMMA 1. For an indecomposable a.p. representation that satisfies the conditions of Theorem 3 the elements of a PF-pair are positive, and the subspaces of invariant vectors and functionals are one-dimensional.

THEOREM 4. Let $T \geqslant 0$ be an a.a.p. representation of an Abelian topological semigroup S in a Krein space $\mathfrak{B}$ that has a vector whose orbit is nonzero. If T is primitive, then $\lim\limits_{s} T(s)\, x = \mu(x)\, h,\ x \in \mathfrak{B}$, where h, μ is a PF-pair for T. Conversely, if the preliminary restrictions on T are satisfied and T is indecomposable, then the fact that it is convergent implies that it is primitive.

COROLLARY 4. If $A \geqslant 0$ is a primitive a.p. operator in a Krein space $\mathfrak{B}$, then the strong $\lim\limits_{n\to\infty} A^n$, exists, and if this limit is nonzero, it is equal to $\mu(\cdot)h$, where h, μ is a PF-pair for A.

3. Let Q be a compactum, $\mathfrak{B} = C(Q)$, and $\mathfrak{C} = C_+(Q)$ the cone of nonnegative continuous functions. In this situation Theorem 3 leads to the following result.

THEOREM 5. Let T be a nonnegative a.p. (or a.a.p. in the Abelian case) representation of a topological semigroup S in the space C(Q), and suppose that the Sushkevich kernel $K_T$ is a group. If $r(T(s)) \equiv 1$, then there is a PF-pair for T.

In fact, $\|T(s)1\| = \|T(s)\| \geqslant r(T(s)) = 1$, that is, the orbit of the function 1 is nonzero.

A representation T in C(Q) is said to be stochastic (Markov) if $T(s)1 \equiv 1$.

Example. Let G be a compact group, Q a compactum, and $F{:}G \to \mathrm{Homeo}(Q)$ the action of G in Q. Putting $(T_F(g)\varphi)(t) = \varphi(F(g^{-1})t)$, we obtain a stochastic representation of G in C(Q).

Two representations $T_1$ and $T_2$ in $C(Q_1)$ and $C(Q_2)$ are said to be positively equivalent if there is an invertible operator $V{:}C(Q_1) \to C(Q_2)$ interlacing them that is nonnegative together with $V^{-1}$. We note that automorphisms of the Krein space C(Q) are described as follows: $(V\varphi) \times (t) = \omega(t)\, \varphi(\gamma(t))$, where $\gamma{:}Q \to Q$ is a homeomorphism, and $\omega > 0$. If $\omega = 1$, and only in this case, V is stochastic. Therefore, any stochastic representation of G in C(Q) is generated by some action of it in Q.

LEMMA 2. For a representation $T \geqslant 0$ in C(Q) to be positively equivalent to a stochastic representation it is necessary and sufficient that there should be an invariant function $h > 0$.

From now on we assume that T satisfies all the conditions of Theorem 5. Let P be its boundary projector. Obviously, $P \geqslant 0$, and only this property is used below. Let us put $\varepsilon = P1$, $E_+ = \{t \mid \varepsilon(t) > 0\}$. We define the support supp P of the projector P as the set of all t such that if $\varphi \in C(Q),\ \varphi \geqslant 0,\ \varphi(t) \neq 0$, then $P\varphi \neq 0$ (this definition can be applied to any nonnegative operator). We now put $E = E_+ \cap \mathrm{supp}\, P$ and identify points in E that are indistinguishable by functions of the form $\hat{P}\varphi = P\varphi/\varepsilon$. We denote the resulting factor space by $\tilde{E}$, and the composition of the operator $\hat{P}$ with the transition to functions on $\tilde{E}$ by $\tilde{P}$.

LEMMA 3. The factor space $\tilde{E}$ is compact. The operator $\tilde{P}$ maps C(Q) onto $C(\tilde{E})$, and $\|\tilde{P}\| \leqslant 1$.

COROLLARY 5. The ordered Banach spaces Im P and $C(\tilde{E})$ are isomorphic.

On the basis of what we have said we can prove the following basic structural theorem.

THEOREM 6. A representation T on its boundary subspace $\mathfrak{B}_I$ is positively equivalent to the stochastic representation generated by the natural action $F_T$ of the Sushkevich kernel $K_T$ on the factor compactum $\tilde{E}$. If T is stochastic, then this equivalence is an isometry.

A representation T is said to be ergodic if the action $F_T$ is transitive and also E = Q (that is, $\varepsilon > 0$, supp P = Q). In the general case $\tilde{E}$ splits into the orbits of $F_T$. Their inverse images in E are called ergodic classes. Every ergodic class is closed in E and is the union of some family of equivalence classes corresponding to the factorization $E \to \tilde{E}$. The latter are called imprimitivity classes; the number of them (or $\infty$, if there are infinitely many of them) is called the index of imprimitivity of the given ergodic class. An ergodic class is said to be primitive if its index of imprimitivity is equal to 1. Finally, an ergodic representation is said to be topologically imprimitive if its unique ergodic class is primitive.

THEOREM 7. For a representation T to be ergodic it is necessary and sufficient that it should be indecomposable.

THEOREM 8. For a representation T to be topologically primitive it is necessary and sufficient that it should be primitive.

We note that indecomposability in C(Q) reduces to the following: $\forall \varphi \geqslant 0$ $(\varphi \neq 0)$ $\forall t \in Q \exists s$: $(T(s)\ \varphi)(t) > 0$.

With the classes described above there are associated ergodic components and components of imprimitivity of the representation itself. Their construction is rather awkward, and we shall not dwell on it here.

The spectral properties of ergodic representations are summarized in the following theorem.

THEOREM 9. Let T be an ergodic representation. Then 1) the unitary weights of T form a group, the corresponding weighted subspaces are one-dimensional, and the moduli of all the weighted functions are proportional to an invariant function h > 0 (in the stochastic case they are constants); 2) the representation T is positively equivalent to a stochastic representation in the same space; 3) if $\chi$ is a unitary weight, then the representation $\chi \otimes T$ is equivalent to T; 4) if S is Abelian, then the representation $T | \mathfrak{B}_1$ is positively equivalent to the representation generated by the regular action of S on the Sushkevich kernel $K_T$. The group of unitary weights coincides with the dual group $K_T^*$.

The spectral theory of stochastic a.p. operators in C(Q) was mainly constructed in [4-6]. These results are covered by the theory developed above, from which there follows the following series of assertions.

THEOREM 10. Let A be an a.p. operator in C(Q), $A \geqslant 0$, $r(A) = 1$. We denote the boundary projector by P. Then the following assertions are true.

1. $A^n | \mathrm{Ker}\, P \underset{n\to\infty}{\to} 0$ (strongly), and $A | \mathrm{Im}\, P$ is positively equivalent to the stochastic operator generated by a homeomorphism $F_A$ of some compactum $\tilde{E}$.
2. There is a PF-pair h, $\mu$ for the operator A; h > 0, $\mu$ > 0.
3. If A is indecomposable, then a) $\tilde{E}$ is a factor compactum of the compactum Q, and the homeomorphism $F_A$ is topologically conjugate to a topologically transitive shift with respect to the group K; b) the unitary discrete spectrum of A is a group isomorphic to K*; c) the eigensubspaces corresponding to the unitary eigenvalues are one-dimensional; the moduli of the corresponding eigenfunctions are proportional to h; d) for every eigenvalue $\lambda$ with $|\lambda| = 1$ the operator $\lambda A$ is similar to A, and thus the spectrum of A is invariant with respect to multiplication by $\lambda$.
4. If A is primitive, then $\lim_{n\to\infty} A^n \varphi = \left(\int_Q \varphi d\mu\right) h$ for all functions $\varphi \in C(Q)$.

The last assertion gives a new approach to the so-called "Ruelle version" of the Perron–Frobenius theorem [7], and is also applied in some related questions in the theory of dynamical systems [8].

LITERATURE CITED


1. K. deLeeuw and I. Glicksberg, "Applications of almost-periodic compatifications," Acta Math., 105, 63-97 (1961).
2. Yu. I. Lyubich, "On the boundary spectrum of contractions in Minkowski spaces," Sib. Mat. Zh., 11, 165-171 (1970).
3. K. Yosida and S. Kakutani, "Operator-theoretical treatment of Markov's process and mean ergodic theorem," Ann. Math., 42, 188-228 (1941).
4. B. Jamison, "Asymptotic behavior of iterates of continuous functions under a Markov operator," J. Math. Anal. Appl., 9, 203-214 (1964).

5. M. Rosenblatt, "Equicontinuous Markov operators," Teor. Veroyatn. Primen., 9, 205-222 (1964).
6. B. Jamison and R. Sine, "Irreducible almost-periodic Markov operators," J. Math. Mech., 18, 1043-1057 (1969).
7. R. Bowen, Methods of Symbolic Dynamics [Russian translation], Mir, Moscow (1979).
8. M. Yu. Lyubich, "On the measure of maximal entropy of a rational endomorphism of the Riemann sphere," Funkts. Anal. Prilozhen., 16, No. 4, 78-79 (1982).

# ASYMPTOTIC OF SOLUTIONS THAT COVER INTEGRAL *O*-SET OF A NONLINEAR SYSTEM

S. K. Norkin

UDC 517.925.34

1. Let us consider the system

$$\alpha(x)\, dy/dx = F(x, y), \tag{1}$$

defined in the domain

$$S = \{(x, y) : x \in ]0;\ a[,\ \|y\| \in [0;\ b[\ \}, \tag{2}$$

where $x \in \mathbb{R}_+$, $(u_1, u_2, u_3) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{n-k-m}$, $n > k + m$, $y = \operatorname{colon}(u_1, u_2, u_3)$; $(\alpha_1, \alpha_2, \alpha_3): \mathbb{R}_+ \to \mathbb{R}_+^3$; $\alpha(x) = \operatorname{diag}(\alpha_1(x), \alpha_2(x), \alpha_3(x))$; $(F_1, F_2, F_3): \mathbb{R}^{1+n} \to \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^{n-k-m}$, $F(x, y) = \operatorname{colon}(F_1(x, y), F_2(x, y), F_3(x, y))$; $a, b \in \mathbb{R}_+$; and $\|\cdot\|$ is the Euclidean norm.

System (1) is assumed to be from the class ($\mathbb{C}$ ; uniqueness) in the domain S, i.e., the vector-valued functions α and F are continuous in the domain S and have a property which ensures the uniqueness of solution of system (1) that satisfies arbitrary initial conditions in S.

Several authors [1-6] have studied the *O*-set $\mathfrak{M} \subset S$ that is covered by the *O*-solutions of system (1), i.e.,

$$\mathfrak{M} = \{(x, y(x)) \in S : \alpha(x)\, y'(x) \equiv F(x, y(x)),\ \lim_{x \to +0} \|y(x)\| = 0\}$$

for the case n = k + m, have separated the "stable" and the "unstable" components $u_1$ and $u_2$ of the vector y, and have also found the conditions under which $\dim \mathfrak{M} = n - k + 1$. In the present article, we find the asymptotic for the *O*-solutions of the set $\mathfrak{M}$ on the basis of the obtained two-sided estimates for the component $u_3$ when n > k + m.

2. We suppose that the vector-valued function F = colon ($F_1$, $F_2$, $F_3$) satisfies the following conditions: The following inequalities are satisfied in the domain S for all x, y = colon ($u_1$, $u_2$, $u_3$):

$$(u_1, F_1(x, y)) \leqslant -L_1(x)\|u_1\|^2 + \lambda_1(x)\|y\|^2, \tag{3}$$

$$(u_2, F_2(x, y)) \geqslant L_2(x)\|u_2\|^2 - \lambda_2(x)\|y\|^2, \tag{4}$$

$$|(u_3, F_3(x, y)) - \lambda_0(x)\|u_3\|^2| \leqslant L_3(x)\|u_3\|^2 + \lambda_3(x)\|y\|^2, \tag{5}$$

where the functions $L_i(x)$ and $\lambda_j(x)$, i = 1, 2, 3 and j = 0, 1, 2, 3, are defined and continuous in the interval ]0; *a*[ and $\lambda_j(x) \geqslant 0$, $L_3(x) + \lambda_3(x) \geqslant 0$.

Let the asymptotic of the function $\alpha_i(x)$ and the functions $L_i(x)$ and $\lambda_j(x)$ be defined by the following equations:

$$\lim_{x \to +0} \alpha_j(x) = 0, \quad j = 1, 2, 3; \tag{6}$$

$$\lim_{x \to +0} \alpha_i(x)/\alpha_3(x) = c_i < +\infty, \quad i = 1, 2; \tag{7}$$

$$L_i(x) = L_i - d_i(x), \quad i = 1, 2; \qquad L_3(x) = L_3 + d_3(x), \tag{8}$$

Odessa Polytechnic Institute. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 36, No. 5, pp. 636-641, September-October, 1984. Original article submitted April 9, 1983.

 0041-5995/85/3605-0478$08.50 