Theorem 9 as in the proof of Theorem 2. The case $\ln p(\omega) = O\{\ln \mu(\omega)\}$ is handled by means of Lemma 4.

Remark 3. The conditions (4) in Theorem 1 are essential. The first of them guarantees, by a theorem of Zygmund [1], the (R, p, α)-summability almost everywhere on series (1). The second condition is also essential: If the function $\lambda(\omega)$ is such that $p(\omega) \ln \ln p(\omega) = O\{\lambda(\omega)\}$, then Lemmas 3 and 5 and condition (2) imply the existence almost everywhere on $[a, b]$ of a finite limit of the function $p(\omega)[f(x) - R_\omega^\alpha(x)]$ as $\omega \to \infty$ and the inequality $p(\omega)\, \Delta_\omega^\alpha(x) \le F(x) \in L^2[a, b]$, and it is impossible to strengthen this result (see [4, pp. 59-60] and [6, p. 15]).

Note also that in the proof of definitiveness of the estimates the choice of the system $\{\varphi_n(x)\}$ and the coefficients of series (1) depends essentially on the functions $v(\omega)$, $\lambda(\omega)$, and $p(\omega)$.

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MEASURE OF SOLENOIDAL ATTRACTORS OF UNIMODAL MAPS OF THE SEGMENT

A. M. Blokh and M. Yu. Lyubich

1. Introduction. We consider the class $\mathfrak{S}_1$ of C^3-smooth maps $f: [0, 1] \to [0, 1]$ of the segment, possessing the following properties:

U1) f has a unique critical point $c \in (0, 1)$ and this point is an extremum;

U2) in the neighborhood of the point c we have the inequalities

$$B_1 |x - c|^n \leqslant |f'(x)| \leqslant B_2 |x - c|^n;$$

U3) outside the extremum, f has a negative Schwarzian:

$$Sf = f'''/f' - (3/2)(f''/f')^2 < 0.$$

The maps of class $\mathfrak{S}_1$ present great interest from the point of view of the theory of dynamical systems (see [1]).

By a solenoidal attractor (or, simply, a solenoid) of the transformation f we mean a totally disconnected, invariant, compact $S \subset [0, 1]$, having the structure

$$S = \bigcap_{m=1}^{\infty} \bigcup_{k=0}^{p_m - 1} f^k I_m,$$

Arctic and Antarctic Scientific Research Institute Goskomgidrometa SSSR.

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where $I_1 \subset I_2 \subset \ldots \ni c$, $p_m \to \infty$, the segments $f^k I_m$ for $k = 0, 1, \ldots, p_m - 1$ are pairwise disjoint, and $f^{p_m} I_m \subset I_m$ (i.e., I_m is a periodic segment with minimal period p_m). We mention that p_{m+1} is divisible by p_m, the restriction $f|S$ is a homeomorphism, and $S = \omega(c) \ni c$, where $\omega(c)$, as usually, denotes the limit set of the orbit $\{f^m c\}_{m=0}^{\infty}$. For more details on solenoids, see [2].

With the aid of the kneading theory [3] one can show that already in the family $x \to a(x-1)$ one finds solenoids with arbitrary sequences of periods $\{p_m\}_{m=1}^{\infty}$ (for which p_{m+1} is divisible by p_m). Binary solenoids S(for which $p_{m+1}/p_m = 2$, $p_1 = 2$, $m = 1, 2, \ldots$) are usually called Feigenbaum attractors. A vast literature, connected with the renormalization-group, has been devoted to them (see [4]). Non-binary solenoids have been investigated in a significantly more deficient manner.

Let λ be the Lebesgue measure on [0, 1]. With the aid of Feigenbaum's universal law, it is easy to show that $\lambda(S) = 0$ for a binary solenoid S (moreover, one can estimate its Hausdorff [4, 5]). Guckenheimer has proved a result on the measure of a binary solenoid without using the Feigenbaum universality [6]. The purpose of this paper is the proof of an analogous result for arbitrary solenoids.

THEOREM. Let S be a solenoidal attractor of a transformation $f \in \mathfrak{S}_1$. Then $\lambda(S) = 0$.

Remark 1. In the case when $\sup(p_{n+1}/p_n) < \infty$, the theorem can be refined: the Hausdorff dimension of a corresponding solenoid is less than 1.

Remark 2. The theorem can be generalized also in the following manner. We assume that the invariant set A is nowhere dense, while $f|A$ is injective. Then $\lambda(A) = 0$.

2. Notations. By $[a, b]$ we denote the (closed) interval with endpoints a, b, without assuming that $a < b$. For a point x we set $x_n = f^n x$ (similarly a_n, c_n, ...). In particular, $x_0 \equiv x$. In the neighborhood of the extremum c we define an involution $\tau\colon x \to x'$ in the following manner: $f(x') = f(x)$. By virtue of property U2, τ satisfies a Lipschitz condition with some constant L. We shall say that the point a is situated closer to the extremum than b if $a \in (b, \tau(b))$. We shall write $A \subset B$, if $A \subset B$ or $\tau(A) \subset B$.

By $H_n(x)$ we denote the maximal monotonicity interval of the function f^n, containing the point x. The endpoints of such an interval are points of the set $\bigcup_{k=0}^{n-1} f^{-k} c \cup \{0, 1\}$.

By Guckenheimer's theorem on the absence of homtervals [7], if f has a solenoid, then

$$\max_{x \in [0,1]} \lambda(H_n(x)) \to 0 \quad (n \to \infty). \tag{1}$$

We set $M_n(x) = f^n H_n(x)$. By $M_n^{\pm}(x)$ we denote the intervals into which the point x_n divides $M_n(x)$, while by $H_n^{\pm}(x)$ the corresponding intervals in $H_n(x)$, $f^n H_n^{\pm}(x) = M_n^{\pm}(x)$. Moreover, if x_n is near to c, then by $M_n^{+}(x)$ we shall denote the interval which is farther from c than x.

The notation $\varphi\colon [a_0, a_1, \ldots, a_k] \to [b_0, b_1, \ldots, b_k]$ will be used in the case when the sequence $\{a_i\}_{i=0}^{k}$ is monotone, the restriction $\varphi|[a_0, a_k]$ is monotone, and $\varphi(a_i) = b_i$ $(i = 0, \ldots, k)$.

If the set $A \subset [0, 1]$ is measurable, $I = [a, b]$ is an interval, then

$$\rho_A(I) \equiv \rho_A([a, b]) = \lambda(A \cap I)/\lambda(I)$$

will denote the density of the set A in the interval I.

3. Lemmas on Distortion.

LEMMA 1 (on distortion). Let $f\colon [0, 1] \to [0, 1]$ be a C^3-smooth transformation with a finite number of critical points, in the neighborhood of which property U2 holds. Let I, J be two abutting closed intervals, not containing critical points. If $\lambda(I) \le \lambda(J)$, then

$$\frac{\lambda(fI)}{\lambda(fJ)} : \frac{\lambda(I)}{\lambda(J)} \leqslant A(f).$$

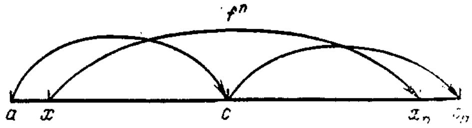

Fig. 1

The proof of this elementary fact is left to the reader.

We consider now that class $\mathfrak{S}_0$ of C^3-smooth surjective maps φ: [0, 1] → [0, 1] with a negative Schwarzian, having no critical points. The following important result, due to Guckenheimer, is analogous to Koebe's well-known distortion theorem for univalent functions (see [8]).

Koebe's Property [6]. Let $f \in \mathfrak{S}_0$, $\delta > 0$. If $\varphi(x)$, $\varphi(y) \in [\delta, 1-\delta]$, then $|\varphi'(x)|/|\varphi'(y)| \leq B(\delta)$, where $B(\delta)$ does not depend on φ.

From Koebe's property one derives easily the following two lemmas.

LEMMA 2 (on distortion). Let $\varphi \in \mathfrak{S}_0$, let $[0, 1] = H^- \cup K \cup H^+$ be a partition of the segment [0, 1] into a union of three intervals. There exists a function γ: $\mathbf{R}_+ \to \mathbf{R}_+$, independent on φ, such that

$$\frac{\lambda(\varphi H^{\pm})}{\lambda(\varphi K)} \geqslant \alpha \Rightarrow \frac{\lambda(H^{\pm})}{\lambda(K)} \geqslant \gamma(\alpha).$$

LEMMA 3 (on distortion). Let $\varphi \in \mathfrak{S}_0$, let $[0, 1] = H \cup K$ be a partition of the segment [0, 1] into a union of two intervals, and let A be a measurable subset of [0, 1]; then $\lambda(\varphi H)/\lambda(\varphi K) \geq \nu > 0$, $\rho_{\varphi A}(\varphi K) \geq \alpha \Rightarrow \rho_A(K) \geq \beta(\nu, \alpha)$, where the function $\beta(q, \alpha)$ does not depend on φ.

4. Fundamental Lemmas. In this section we shall assume that the transformation $f \in \mathfrak{S}_1$ does not have limit cycles (i.e., attracting some interval). In particular, this assumption holds if f has a solenoid.

LEMMA 1 (see Fig. 1). Suppose that the point a is such that $f^n|[a, c]$ is monotone and $a_n = c$. We assume that there exists a point $x \in (a, c)$ such that x_n is situated closer to c than the point x itself and than all points c_m (m = 1, 2, ..., n). Then $|x_n - c_n|/x_n - c| \geq q(f) > 0$.

Proof. For the sake of definiteness, we shall assume that $a < c < c_n$. We assume, reasoning by contradiction, that

$$|x_n - c_n|/|x_n - c| < q, \tag{2}$$

where the value of q will be defined below.

We start the proof with the inductive construction of a certain sequence of indices $0 = k(0) < k(1) < \ldots < k(j) < n$. Assume that k(i) has been already defined in such a manner that $f^{n-k(i)}|[a_{k(i)}, c]$ is monotone. If $n - k(i) \leq 3$, then we set j = i and we conclude the construction; otherwise we consider the interval $[a_{k(i)+3}, c_3]$ and we imbed it into the maximal interval $[a_{k(i)+3}, d^{i+1}]$, terminating at $a_{k(i)+3}$, on which the mapping $f^{n-k(i)-3}$ is monotone. We mention that the selection of the point c_3 is connected with the fact that, obviously, it is not an endpoint of the segment [0, 1].

The point d^{i+1} is either an endpoint of the segment [0, 1] or the f^m-preimage of the extremum c for some $m \in [0, n - k(i) - 3)$. The point c_3 does not have any of these two properties. Consequently, $d^{i+1} \neq c_3$. If d^{i+1} is the f^m-preimage of the point c and, moreover, m > 0, then we set

$$k(i+1) = k(i) + 3 + m. \tag{3}$$

Moreover, $[a_{k(i+1)}, c] = f^m[a_{k(i)+3}, d^{i+1}]$ and, consequently, the mapping $f^{n-k(i+1)}$ is monotone on $[a_{k(i+1)}, c]$. Finally, if $d^{i+1} \in \{0, 1\}$, then we set j = i and we conclude the construction. Since in this case $f^{n-k(j)}|[c_3, b]$ is monotone for some $b \in \{0, 1\}$, it follows that the exponents $n - k(j)$ are uniformly bounded. Thus,

$$| c_{n-k(j)} - c | \geqslant \varkappa (f) > 0, \tag{4}$$

where κ depends only on f.

We describe the mutual disposition of the various points, occurring in this construction (see Fig. 2). We have

$$[x_n, c_{n-k(i)}] = M^{+}_{n-k(i)} (x_{k(i)}).$$

Consequently, we have the property

P1) $x_n < c_n < c_{n-k(1)} < \dots < c_{n-k(j)}$.

Now we set $(i) = k(i) - k(i-1)$ and we verify

P2) $c_{\ell}(i) \in (a_{k(i)}, c)$.

Indeed, $(a_{k(i)}, c) = f^{(i)-3}(a_{k(i-1)+3}, d^i) \subset f^{(i)-3}(a_{k(i-1)+3}, c_3] = (a_{k(i)}, c(i)]$.

P3) $a_{k(i)}$ lies closer to c than $c_{n-k(i)}$. Indeed, $f^{n-k(i)}$ maps monotonically the segment $[a_{k(i)}, c]$ onto $[c, c_{n-k(i)}]$. If $[c, c_{n-k(i)}] \subset [a_{k(i)}, c]$, then one of the segments $[a_{k(i)}, c]$ or $\tau[a_{k(i)}, c]$ is periodic. But then this segment would contain a limit cycle, in spite of the assumption.

<u>Remark.</u> Applying similar considerations to the mapping

$$f^{l(i)} : [a_{k(i-1)}, c] \to [a_{k(i)}, c_{l(i)}] \subset [a_{k(i)}, c],$$

we can see that the following property holds.

P4) $[a_{k(i-1)}, c] \subset [a_{k(i)}, c]$ (i.e., the points $a_{k(i)}$ move off from c with the increase of i).

Finally, according to the assumption of the lemma, we have

P5) the point x and all $c_l(i)$ lie farther from c than x_n.

Now we prove, by induction on i = 0, 1, ..., j, the inequalities

$$| c_{n-k(i+1)} - c_{n-k(i)} | \leqslant | c_{n-k(i)} - c_{n-k(i-1)} | /2, \tag{5_i}$$

assuming that $c_{n-k(-1)} \equiv x_n$. We assume that (5_m) holds for m = 0, ..., i − 1, while (5_i) does not hold. We consider (taking into account P1) the monotone mapping

$$f^{n-k(i)-3}: [a_{k(i)+3}, c_{l(i)+3}, c_3, d^{i+1}] \to [c, c_{n-k(i-1)}, c_{n-k(i)}, c_{n-k(i+1)}].$$

From the validity of (5_{i-1}) and the violation of (5_i) there follows the premise of the second distortion lemma for $\alpha = 1/2$. Consequently,

$$| a_{k(i)+3} - c_{l(i)+3} | \geqslant \gamma (1/2) \, | c_{l(i)+3} - c_3 |. \tag{6}$$

Now we consider the monotone mapping

$$f^3: [a_{k(i)}, c_{l(i)}, c] \to [a_{k(i)+3}, c_{l(i)+3}, c_3].$$

The first distortion lemma, taking into account (6), implies

$$| a_{k(i)} - c_{l(i)} | \geqslant A^{-1}\gamma (1/2) \, | c_{l(i)} - c |, \tag{7}$$

where A is some constant, depending only on f.

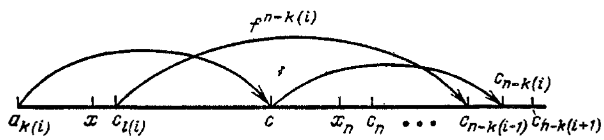

Fig. 2

On the other hand, from P2, P3, P5 there follows the inclusion

$$[a_{k(i)}, c_{l(i)}] \subset [x_n, c_{n-k(i)}].$$

Consequently,

$$\frac{|a_{k(i)} - c_{l(i)}|}{|c_{l(i)} - c|} \leqslant L^2 \frac{|x_n - c_{n-k(i)}|}{|x_n - c|}. \tag{8}$$

It remains to estimate the right-hand side of (8). For this we note that from (5_m), m = 0, 1, ..., i − 1, and (2) there follows

$$|c_{n-k(m)} - c_{n-k(m-1)}| \leqslant 2^{-m} \cdot |x_n - c_n| \leqslant q \cdot 2^{-m} |x_n - c| \quad (m = 0, 1, \ldots, i). \tag{9_m}$$

The summation of (9_m) gives

$$|c_{n-k(i)} - x_n| \leqslant 2q\,|x_n - c|. \tag{10_i}$$

From (8), (10) we conclude

$$|a_{k(i)} - c_{l(i)}) \leqslant 2qL^2\,|c_{l(i)} - c|.$$

But this inequality contradicts (7) for

$$q < q\,(j) \equiv L^{-2} \cdot \gamma\,(1/2)/2A.$$

Thus, in order to prove (5), it remains to verify the induction base (i = 0). With the aid of considerations, entirely similar to those applied to f^3: $[a, x, c] \to [a_3, x_3, c_3]$ and f^{n-3}: $[a_3, x_3, c_3, d^1] \to [c, x_n, c_n, c_{n-k(1)}]$, we obtain that $|a - x| \geq A^{-1}\gamma(1/2)|x - c|$ (this is the analogue of formula (7)) and $[a, x] \subset_\tau [x_n, c_n]$ (this is the analogue of the inclusion $[a_{k(i)}, c_{\ell(i)}] \subset [x_n, c_{n-k(i)}]$). For $q < L^{-2}\gamma(1/2)/2A$ this implies a contradiction, proving the induction base.

In order to conclude the proof of Lemma 1 it remains to note that from (5_i) for i = 0, 1, ..., j there follows the inequality (10_j), which, in turn, implies $|c_{n-k(j)} - c| \leq (2q + 1)|a - c|$. For $a \to c$ the last inequality contradicts (4). Lemma 1 is proved.

LEMMA 2. Assume that the point $x \in [0, 1]$ is such that x_n is situated closer to c than all points c_m (m = 1, ..., n), x_r (r = 0, ..., n − 1).

Then: a) $M_n^-(x) \subset [\tau(x_n), x_n]$; b) $\lambda(M_n^+(x)) \geq d(f) \cdot |x_n - \tau(x_n)|$.

Proof. Statement a) follows from the assumptions of the lemma since $M_n^-(x) = [c_m, x_n]$ for some $m \in [1, n]$.

b) We consider the moment $t \in [1, n)$ for which $f^t H_n^+(x)_- = [x_t, c]$. By virtue of a) there exists a point $a \in f^t H_n^-(x)$, for which $f^{n-t}a = c$. Applying Lemma 1 to the mapping f^{n-t}: $[a, x_t, c] \to [c, x_n, c_{n-t}]$, we obtain the estimate $\lambda(M_n^+(x)) \geq q|x_n - c|$. Making use of the Lipschitz property of the involution τ, we obtain the required estimate with $d(f) = q(L + 1)^{-1}$.

5. Proof of the Theorem. We can assume that the periodic segments I_m, defining the solenoid (see Introduction) are symmetric with respect to c, i.e., $\tau(I_m) = I_m$ (otherwise, they can be symmetrized by considering $I_m \cup \tau(I_m)$).

We assume that $\lambda(S) > 0$. We consider the density point $x \in S$, for which $x_n \neq c_m$ ($n, m \in \mathbf{N}$). For any $m \in \mathbf{N}$ there exists $n < p_m$ such that $x_n \in I_m$ and, moreover, selecting a large m, we can assume that $n \neq 0$. Consequently, x_n is situated closer to c than all the points c_l ($l = 1, \ldots, p_m - 1$) and all the points x_r (r = 0, 1, ..., n − 1). Applying Lemma 2b), we obtain $\lambda(M_n^+(x)) \geq \nu|x_n - \tau(x_n)|$. By Lemma 2a there exists an interval $\tilde{H}_n^- \subset H_n^-(x)$ such that $f^n\tilde{H}_n^- = [\tau(x_n), x_n]$. Let $\rho(I) \equiv \rho_{[0,1]\setminus S}(I)$ denote the density of the set $[0, 1]\setminus S$ in the interval I. Then for large n we have $\rho[\tau(x_n), x_n] \geq (1 + L)^{-1}$ since f is injective on $[\tau(x_n), x_n] \cap S$. Applying the third distortion lemma to the mapping f^n: $\tilde{H}_n^- \cup H_n^+ \to$

$[\tau(x_n), x_n] \cup M_n^+$, we obtain $\rho(\tilde{H}_n^-) \geq \beta(\nu, (L+1)^{-1}) > 0$. But this inequality contradicts the fact that x is a density point for S. The theorem is proved.

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CONSTRUCTION OF A FUCHSIAN EQUATION FROM A MONODROMY REPRESENTATION

A. A. Bolibrukh

It is known that for any homomorphism

$$\chi\colon \pi_1(CP^1 \setminus D, z_0) \to GL(p; C) \tag{1}$$

of the fundamental group of the complement of a set $D = \{a_1, \ldots, a_n\}$ of points of the Riemann sphere CP^1 into the group of complex nondegenerate matrices of order p one can construct a Fuchsian equation

$$y^{(p)} + q_1(z)\, y^{(p-1)} + \ldots + q_p(z)\, y = 0 \tag{2}$$

with given monodromy (1), whose set D' of singular points coincides with $D \cup \{b_1, \ldots b_m\}$ (see [1]). The supplementary singular points $b_1, \ldots, b_m$ have no contribution to the monodromy and are referred to as "false" (or apparent) singular points. For an irreducible representation (1) the number of such points can be estiamted as follows.

Given the representation (1), one constructs in standard manner a vector bundle F' over $CP^1 \setminus D$ with structure group GL(p; C) [2]. Let F denote the Manin continuation (see [1, p. 95]) of this bundle to all of CP^1 (see also [3]). By the Birkhoff–Grothendieck theorem [4],

$$F \cong \mathcal{O}(-k_1) \oplus \ldots \oplus \mathcal{O}(-k_p), \tag{3}$$

where $k_1 \geq \ldots \geq k_p$ and O(-r) is the r-th power of the Hopf bundle O(−1) over CP^1. Let denote the number of first equal numbers $k_1, \ldots k_p$ ($k_1 = \ldots = k_l$).

<u>THEOREM 1.</u> For any irreducible representation (1) there exists a Fuchsian equation (2) with given monodromy (1), the number m of false singular points of which satisfies the inequality

$$m \leqslant [(n-2)\, p\, (p-1)]/2 - \textstyle\sum_{i=1}^{p} (k_1 - k_i) + 1 - l \tag{4}$$

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