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Parapuzzle of the Multibrot set and typical dynamics of unimodal maps

JEMS

To the memory of Adrien Douady

Abstract. We study the parameter space of unicritical polynomials $f_c : z \mapsto z^d + c$. For complex parameters, we prove that for Lebesgue almost every c, the map f_c is either hyperbolic or infinitely renormalizable. For real parameters, we prove that for Lebesgue almost every c, the map f_c is either hyperbolic, or Collet–Eckmann, or infinitely renormalizable. These results are based on controlling the spacing between consecutive elements in the "principal nest" of parapuzzle pieces.

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1. Introduction

In this paper we are concerned with the dynamics of *unicritical* polynomials

$$f_c : z \mapsto z^d + c, \tag{1.1}$$

where $d \geq 2$, both on the real line (for real values of c) and on the complex plane (in the general case).

Until recently, the dynamical theory of the quadratic family ($d = 2$) was developed much deeper than its counterpart for the higher degree unicritical polynomials (see [H, M2, L1, S], [L3]–[L5], [AM1]). The reason was that the quadratic maps possess some

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very special geometric features that distinguish them from their higher degree cousins. Recently, new tools have been developed [KL1, KL2, AKLS] that opened an opportunity to bring the higher degree case to the same level of maturity as the quadratic one.[1] In this paper that deals with the at most finitely renormalizable case, combined with forthcoming notes (see [AL]) dealing with the infinitely renormalizable case, we intend to accomplish this goal.

For $d \geq 2$ fixed, let $\mathcal{M} = \mathcal{M}_d = \{c \in \mathbb{C} :$ the Julia set of f_c is connected$\}$ be the corresponding *Multibrot set*. The dynamics when $c \notin \mathcal{M}$ is always trivial, so we are mostly concerned with the description of the dynamics for $c \in \mathcal{M}$. When d is odd, the real dynamics is trivial for all $c \in \mathbb{R}$, since f_c is a homeomorphism, so when discussing real dynamics we will always assume that d is even. In this case, for $c \in \mathcal{M} \cap \mathbb{R}$, f_c is a unimodal map.

In what follows, various properties of a map f_c will also be attributed to the corresponding parameter c. For a real c, the map f_c (and the parameter c itself) are called

- *regular* if f_c has an attracting periodic orbit;
- *infinitely renormalizable* if there exist periodic intervals of an arbitrarily large (minimal) period;
- *Collet–Eckmann* if there exist $C > 0$ and $\lambda > 1$ such that

$$|Df^n(c)| \geq C\lambda^n, \quad n = 0, 1, 2, \ldots.$$

Such a map has a unique absolutely continuous invariant measure with strong statistical properties.

We can now formulate our main result on the real dynamics:

Theorem 1.1. *For almost every $c \in \mathcal{M}_d \cap \mathbb{R}$, the map f_c is either regular, or Collet–Eckmann, or infinitely renormalizable.*

Remark 1.1. In [MN], Martens and Nowicki described a bigger class of unimodal maps that have an absolutely continuous invariant measure. In [L4], it was proved that for almost every real $c \in \mathcal{M}_2$, the quadratic polynomial f_c is either regular, or Martens–Nowicki, or infinitely renormalizable. The Martens–Nowicki property was then replaced in [AM1] with the much stronger Collet–Eckmann property, thus providing us with Theorem 1.1 in the quadratic case.

Remark 1.2. With Theorem 1.1 in hand, we can go further, in the same way as in the quadratic case, to show that the whole fine statistical description of the dynamics of real quadratic maps [AM1], [AM3] is valid in the higher degree case as well.

Remark 1.3. In the forthcoming notes, the above result will be complemented by showing that the set of infinitely renormalizable parameters in $\mathcal{M}_d \cap \mathbb{R}$ has zero Lebesgue measure. (In the quadratic case, this was proved in [L5].)

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[1] See also [Sm, KSS, BSS2] for recent advances in the higher degree case that use different tools.

To stress the difference between the quadratic and the higher degree cases, let us mention one consequence of Theorem 1.1. Recall that a *wild attractor* for a unimodal map is a measure-theoretic attractor (in the sense of Milnor [M1]) which is not a topological attractor. There are no wild attractors in the quadratic family [L2], but they do exist for a sufficiently high even criticality d [BKNS]. Moreover, if d is large enough, the set of parameters $c \in \mathcal{M}_d \cap \mathbb{R}$ for which the wild attractor exists contains a Cantor set.

Corollary 1.2. *For any even criticality d, the set of parameters $c \in \mathcal{M}_d \cap \mathbb{R}$ for which the wild attractor exists has zero Lebesgue measure.*

Remark 1.4. One can show (using the estimates of this paper) that for large even d, the set of parameters $c \in \mathcal{M}_d \cap \mathbb{R}$ for which the wild attractor exists has positive Hausdorff dimension.

Remark 1.5. In [BSS2], it was proved that for almost every $c \in \mathcal{M}_d \cap \mathbb{R}$, the map f_c admits a physical measure,[2] which is either supported on an attracting periodic orbit, or absolutely continuous (but f_c is not necessarily Collet–Eckmann), or is supported on a uniquely ergodic Cantor set coinciding with the postcritical set (this possibility contains strictly the infinitely renormalizable case and the case of wild attractors).

The set of non-regular, non-infinitely renormalizable real parameters does have positive Lebesgue measure [J], [BC]. The situation is quite different for complex parameters:

Theorem 1.3. *For almost any $c \in \mathbb{C}$, the map $f_c : z \mapsto z^d + c$ is either hyperbolic or infinitely renormalizable.*

This result was proved for quadratic maps by Shishikura (see a sketch in [S]; see also [AM2] for a proof closer to this paper). We actually prove the following estimate:

Theorem 1.4. *Let f_c be a non-renormalizable map with all fixed points repelling.[3] Then c is not a density point of $\mathcal{M}_d$.*

Again, the last two results are more surprising in the case $d > 2$: while all the quadratic maps in question have the Julia set of zero area ([L1], [S]), it is conceivable that there exist higher degree non-renormalizable unicritical polynomials with the Julia set of positive measure. So, in the quadratic case the phase-parameter dictionary works in the natural way: zero area of Julia sets of the class of maps under consideration translates into zero area of the corresponding set of parameters. On the other hand, in the higher degree case, the phase-parameter correspondence is more subtle.

An important special feature of the quadratic maps essentially exploited in the previous work is the *decay of geometry of the principal nest* (see [L3, L4, AM1]). In this paper we demonstrate that, though in the higher degree case this property fails in general, it is satisfied for *almost all* non-regular non-infinitely renormalizable parameters. This is the key to all of the above results.

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[2] A measure μ is called *physical* if the Birkhoff averages of Lebesgue almost all orbits converge to μ.

[3] The result still holds under the assumption that f_c is not infinitely renormalizable, and has all periodic orbits repelling. The argument for this generalization (which is more subtle than the usual application of the renormalization operator) is indicated in Remark 6.1.

1.1. Notations

- $\mathbb{D}_r = \{z : |z| < r\}$, $\mathbb{D} = \mathbb{D}_1$, $\mathbb{T} = \partial\mathbb{D}$.
- If S is a hyperbolic Riemann surface, let dist_S be the hyperbolic metric in S (with the usual normalization, so that in the upper-half plane $\mathbb{H}$ we have $\mathrm{dist}_{\mathbb{H}}(i, ai) = |\log a|$, $a > 0$). The diameter of a subset $X \subset S$ with respect to dist_S will be denoted $\mathrm{diam}_S X$.
- $K(f)$ is the *filled Julia set* of f.
- $J(f) = \partial K(f)$ is its *Julia set*.
- $\mathrm{Dil}(h)$ stands for the *dilatation* of a quasiconformal map h.
- $\mathrm{mod}(A)$ stands for the modulus of the annulus A.
- *Pullbacks* of an *open* topological disk V under f are connected components of $f^{-1}(V)$.
- *Pullbacks* of a *closed* disk V are closures of pullbacks of int V.

2. Holomorphic motions and a phase-parameter lemma

Let $\Lambda \subset \mathbb{C}$ be a Jordan disk. A *holomorphic motion* over Λ (with base point $\lambda_0 \in \Lambda$) of some set $Z \subset \mathbb{C}$ is a family of injective maps $h_\lambda : Z \to \mathbb{C}$, $\lambda \in \Lambda$, such that for every $z \in Z$, the "trajectory" (or the "orbit") $\lambda \mapsto h_\lambda(z)$ is holomorphic in λ and $h_{\lambda_0}(z) = z$. Given such a holomorphic motion, we let $Z_\lambda = h_\lambda(Z)$.

The central result in the theory of holomorphic motions is the λ-*lemma*. It consists of two parts: extension and quasiconformality. The Extension Theorem (in its strongest version due to Słodkowski [Sl]) says that a holomorphic motion $h_\lambda : Z \to \mathbb{C}$ over a Jordan disk Λ can always be extended to a holomorphic motion $\hat{h}_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole plane over the same Λ. The Quasiconformality Theorem (Mañé–Sad–Sullivan [MSS]) states that each $\hat{h}_\lambda$ is quasiconformal and

$$\log \mathrm{Dil}(h_\lambda) \leq \mathrm{dist}_\Lambda(\lambda_0, \lambda).$$

We say that a holomorphic motion $h_\lambda : Z \to \mathbb{C}$ is *continuous up to the boundary* if the map $(\lambda, z) \mapsto h_\lambda(z)$ extends continuously to $\bar{\Lambda} \times Z$. A holomorphic motion h_λ of a Jordan curve T over Λ which is continuous up to the boundary will be called a *tubing* of T over Λ. Under these circumstances, a *diagonal* to the tubing is a holomorphic function ψ in a neighborhood of $\bar{\Lambda}$ satisfying the following properties:

(D1) For $\lambda \in \Lambda$, $\psi(\lambda)$ belongs to the bounded component of $\mathbb{C} \setminus T_\lambda$, and for $\lambda \in \partial\Lambda$, $\psi(\lambda) \in T_\lambda$.
(D2) For any $\lambda \in \partial\Lambda$, the point $\psi(\lambda)$ has only one preimage $\gamma(\lambda) \in T$ under $h_\lambda|T$.
(D3) The holomorphic motion of a neighborhood of $\gamma(\lambda)$ in T admits an extension over a neigborhood of λ.
(D4) The graph of ψ crosses the orbit of $\gamma(\lambda)$ transversely at $\psi(\lambda)$.
(D5) The map $\gamma : \partial\Lambda \to T$ has degree 1.

Remark 2.1. Note that properties (D3) and (D4) imply that $\gamma : \partial\Lambda \to T$ is continuous, so that (D5) makes sense.

Given a set Z contained in the closed Jordan disk bounded by T, we say that a holomorphic (and continuous up to the boundary) motion H_λ of Z over Λ *fits the tubing* of T if for every $\lambda \in \overline{\Lambda}$, we have $H_\lambda(z) = h_\lambda(z)$ for $z \in Z \cap T$, while $H_\lambda(z) \notin h_\lambda(T)$ for $z \in Z \setminus T$.

Lemma 2.1. *Let $h_\lambda : Z \to \mathbb{C}$ be a holomorphic motion over a Jordan disk Λ continuous up to the boundary that fits a tubing of a Jordan curve T. Let ψ be a diagonal to this tubing. Then for each point $z \in Z$ there exists a unique parameter $\lambda = \chi(z) \in \Lambda$ such that $h_\lambda(z) = \psi(\lambda)$. The map $\chi : Z \to \overline{\Lambda}$ is continuous and injective. Moreover, if $z \in \mathrm{int}\, Z$ and $h_{\chi(z)}$ is locally K-quasiconformal at z then χ is locally K-quasiconformal at z.*

Proof. Without loss of generality, we can assume that Λ is the unit disk with base point at the origin. By assumptions (D3)–(D4), $\gamma : \partial \Lambda \to T$ is a local homeomorphism. By (D5), it has degree 1, so that it is a homeomorphism. Letting $\chi | (Z \cap T) = \gamma^{-1}$, we see that the first assertion is valid for $z \in Z \cap T$.

Let $z \in Z \setminus T$. By applying an appropriate family of affine changes of variable, we can reduce the considerations to the case when $h_\lambda(z) = 0$, $\lambda \in \overline{\mathbb{D}}$.

Let us consider a torus $\mathbb{T}_0^2 = \mathbb{T} \times T \subset \mathbb{T} \times \mathbb{C}$. Let us deform it in $\mathbb{T} \times \mathbb{C}$ as follows:

$$H_r : \mathbb{T}_0^2 \to \mathbb{T}_r^2, \ (\lambda, z) \mapsto (\lambda, h_{r\lambda}(z)), \quad \lambda \in \mathbb{T}, \ z \in T, \ 0 \le r \le 1.$$

Since the motion of the origin fits the tubing of T, the deformations never cross the core circle $\mathbb{T} \times \{0\}$.

Let us consider a family of curves $\psi_r : \mathbb{T} \to \mathbb{C}$, $\psi_r(\lambda) = h_{r\lambda} \circ \gamma(\lambda)$, $0 \le r \le 1$. Note that the graph of ψ_r is a curve in $\mathbb{T}_r^2$ obtained by applying the homotopy H_r to the graph of ψ_0. Since the $\mathbb{T}_r^2$ are disjoint from the core circle, the curves ψ_r never pass through the origin and hence have the same winding number around it. Since $\psi_0 = \gamma$, by (D5) this winding number is equal to 1. But $\psi_1 = \psi | \mathbb{T}$ by definition of γ (D2). By the Argument Principle, ψ has a single root in $\mathbb{D}$, which proves the first assertion.

Any point $\lambda \in \Lambda$ has at most one preimage under χ since the maps h_λ are injections. A point $\lambda \in \partial \Lambda$ has only one preimage $\chi^{-1}(\lambda) = \gamma(\lambda)$ by (D2) and the assumption that the motion of Z fits the tubing of T. The graph of χ is the set of solutions (z, λ) of $h_\lambda(z) = \psi(\lambda)$, which is clearly closed in $Z \times \overline{\Lambda}$, so χ is continuous.

Local quasiconformality of χ follows from the λ-lemma (see Corollary 2.1 of [L4]). □

We will often encounter the situation when Z contains an annulus A, and we want to obtain a lower bound on $\mathrm{mod}(\chi(A))$. A trivial bound

$$\mathrm{mod}(\chi(A)) \ge K^{-1} \mathrm{mod}(A), \quad \text{where} \quad K = \sup_{\lambda \in \Lambda} \mathrm{Dil}(h_\lambda | A),$$

will sometimes be sufficient. However, since the dilatation of the holomorphic motion can blow up as $\lambda \to \partial \Lambda$, it will not cover all of our needs. Then we will make use of the following generalization of Corollary 4.5 of [L4].

Lemma 2.2. *Under the above circumstances, let $X \subset Z \setminus T$, let U_λ be the bounded component of $\mathbb{C} \setminus T_\lambda$, and let $U = U_{\lambda_0}$. Then:*

(1) *There exists a $\delta_0 > 0$ such that if $\mathrm{diam}_{U_\lambda} X_\lambda < \delta \leq \delta_0$ for every $\lambda \in \Lambda$ then $\mathrm{diam}_\Lambda \chi(X) < \epsilon(\delta)$,[4] where $\epsilon(\delta) \to 0$ as $\delta \to 0$.*

(2) *Assume that X is connected and $\mathrm{diam}_U X \leq M$. Assume also that for some $K > 1$ and for every $\lambda \in \Lambda$, the map $h_\lambda : X \to \mathbb{C}$ extends to a K-qc homeomorphism $U \to U_\lambda$. Then $\mathrm{diam}_\Lambda \chi(X) \leq C = C(M, K)$.*

Proof. (1) It is enough to consider the case where X consists of two points. By a holomorphic change of coordinates $(\lambda, z) \mapsto (\lambda, \phi_\lambda(z))$ where $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ is affine, we may assume that $X_\lambda = \{0, 1\}$ for all $\lambda \in \Lambda$. If $\mathrm{diam}_{U_\lambda} X_\lambda$ is small, then $\mathbb{D}_{2R} \subset U_\lambda$ for some large $R > 1$ and all $\lambda \in \Lambda$. Let $\tilde{h}$ be the holomorphic motion of $T \cup \mathbb{D}_R$ obtained by setting $\tilde{h}_\lambda(z) = h_\lambda(z)$ for $z \in T$ and $\tilde{h}_\lambda(z) = z$ for $z \in \mathbb{D}_R$. Notice that $\tilde{h}$ also fits the tubing and $\tilde{\chi}(X) = \chi(X)$.

Since $\tilde{h}$ is holomorphic in $\mathbb{D}_R$, $\tilde{\chi} = \psi^{-1} : \tilde{W} \to \Lambda$ is also holomorphic on $\mathbb{D}_R$. Hence $\mathrm{mod}(\tilde{\chi}(\mathbb{D}_R \setminus \mathbb{D})) = \frac{1}{2\pi} \log R$ and

$$\mathrm{diam}_\Lambda(\chi(X)) \leq \mathrm{diam}_{\tilde{\chi}(\mathbb{D}_R)} \tilde{\chi}(X) = O(1/R).$$

(2) We will use the uniform equicontinuity of K-qc maps with respect to the hyperbolic metric: For any K-qc map $\phi : S \to \tilde{S}$ between hyperbolic Riemann surfaces,

$$\mathrm{dist}(x, y) < \eta \implies \mathrm{dist}(\phi(x), \phi(y)) < \delta(K, \eta),$$

where $\delta(K, \eta) \to 0$ as $\eta \to 0$.

Let us select $\eta = \eta(K)$ so that $\delta(K, \eta) < \delta_0$, where δ_0 comes from the first statement. We can cover X by $N = N(\eta, M)$ sets $X_1, \ldots, X_N$ of hyperbolic diameter in U bounded by η. Then the first statement is applicable to each X_i, so that $\mathrm{diam}_\Lambda \chi(X_i) < \epsilon_0 = \epsilon(\delta_0)$. Since X is connected,

$$\mathrm{diam}_\Lambda \chi(X) \leq \sum \mathrm{diam}_\Lambda \chi(X_i) < N\epsilon_0,$$

and we are done. $\square$

We will need one lemma on lifting of a holomorphic motion by a family of branched coverings.

Lemma 2.3. *Let $h_\lambda : Z \to Z_\lambda$ be a holomorphic motion over a pointed disk (Λ, λ_0), and let $f_\lambda : U'_\lambda \to U_\lambda$ be a holomorphic family of branched coverings of degree d such that $U_\lambda \supset Z_\lambda$.[5] Let $\Omega \Subset \Lambda$ be an open Jordan disk containing λ_0 such that for $\lambda \in \Omega$, the sets Z_λ do not contain the critical values of f_λ. Then h_λ over Ω lifts by f_λ to a holomorphic motion h'_λ continuous up to the boundary.*

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[4] On the other hand, one can show that the statement is false for large δ.

[5] We assume (as part of the definition of a "holomorphic family") that $\bigcup U_\lambda$ and $\bigcup U'_\lambda$ are open subsets of $\mathbb{C}^2$.

Proof. Each orbit $Z(z) = \{(\lambda, h_\lambda(z)) : \lambda \in \Lambda\} \subset \mathbb{C}^2$ of the motion h_λ lifts to a variety $Y(z) = \{(\lambda, z) : (\lambda, f_\lambda(z)) \in Z(z)\}$ which properly projects to Λ with degree d. Since for $\lambda \in \Omega$, Z_λ does not contain critical values of f_λ, these varieties are unbranched over Ω and hence form a holomorphic motion $h'_\lambda : Y \to Y_\lambda$ over it. All we need to show is that it is continuous up to the boundary of Ω.

It is enough to show that for any compact $K \subset Y$, the family $\{\lambda \mapsto h'_\lambda(y)\}_{y \in K}$ is uniformly equicontinuous over Ω. Let $y_n \in K$, $\sigma_n \subset \Omega$, σ_n an arc of diameter at most $1/n$, and let $B_n = \{h'_\lambda(y_n) : \lambda \in \sigma_n\}$. We must show that the diameter of B_n shrinks to 0. We may assume that $y_n \to y \in K$ and $\sigma_n \to \lambda \in \overline{\Omega}$ in the Hausdorff topology. Then, for any $\epsilon > 0$, for large n, B_n lies within the ϵ-neighborhood of $f_\lambda^{-1}(h_\lambda(f_{\lambda_0}(y)))$. Since $f_\lambda^{-1}(h_\lambda(f_{\lambda_0}(y)))$ has at most d elements and B_n is connected, this implies that B_n has diameter at most $2d\epsilon$, as desired. $\qquad\square$

3. Puzzle and parapuzzle

3.1. Parameter and dynamical Böttcher coordinates

The basic dynamical theory of the unicritical family $z \mapsto z^d + c$ (see [Sc1]) is similar to the basic theory of the quadratic family (see [DH1, M3]). For further reference, we recall here the main objects of the theory and set up notations.

The *(dynamical) Böttcher function* B_c conformally conjugates f_c near ∞ to $z \mapsto z^d$. The *Green function* $G_c = \log |B_c|$ extends harmonically to $\mathbb{C} \setminus K(f_c)$. Its level sets $\{G_c = \xi\}$ are called *(dynamical) equipotentials* $\mathcal{E}_\xi^{\text{dyn}} = \mathcal{E}_\xi^{\text{dyn}}(c)$. They form an invariant foliation with singularities at the precritical points (at each singularity, the equipotential looks locally like the intersection of d lines). Let

$$\Delta_c = \{z : G_c(z) > G_c(0)\}.$$

It is the maximal neighborhood of ∞ saturated by the equipotentials on which the foliation is non-singular.

The gradient lines of G_c coming from infinity are called *(dynamical) external rays*. They form a foliation of $\mathbb{C} \setminus K(f)$ slit along the gradient lines emerging from the critical points of the Green function. The argument ("angle") of B_c is constant on each ray. The ray of angle θ is denoted by $\mathcal{R}_\theta^{\text{dyn}} = \mathcal{R}_\theta^{\text{dyn}}(c)$.

If the Julia set of f is connected, the Böttcher function extends analytically to the whole basin of infinity, $\mathbb{C} \setminus K(f_c)$, and maps it conformally onto $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Otherwise, B_c extends analytically to the domain Δ_c, and maps it conformally onto $\mathbb{C} \setminus \mathbb{D}_{\rho(c)}$, where $\rho(c) = e^{G_c(0)} > 1$. In this case, the function

$$B_\mathcal{M}(c) = B_c(c) \tag{3.1}$$

is well defined and is called the *(parameter) Böttcher function*. It provides us with the Riemann mapping $\mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}$. This *basic relation* between the dynamical and parameter Böttcher coordinates/Riemann mappings is the foundation of the phase-parameter correspondences for the unicritical families of polynomials.

The *(parameter) equipotentials and external rays*, $\mathcal{E}_\xi^{\mathrm{par}}$ and $\mathcal{R}_\theta^{\mathrm{par}}$, are the level sets and the gradient lines of the parameter Green function $G_\mathcal{M}(c) = \log|B_\mathcal{M}(c)|$. They form two (non-singular) orthogonal foliations on $\mathbb{C} \setminus \mathcal{M}$. By basic relation (3.1),

- $c \in \mathcal{E}_\xi^{\mathrm{par}}$ iff $c \in \mathcal{E}_\xi^{\mathrm{dyn}}(c)$;
- $c \in \mathcal{R}_\theta^{\mathrm{par}}$ iff $c \in \mathcal{R}_\theta^{\mathrm{dyn}}(c)$.

(In each line, the first and the last "c" stand for the *parameter*, while the intermediate one stands for the *critical value*.)

Let

$$\mathbf{F} = \{(c, z) \in \mathbb{C}^2 : z \in \mathbb{C} \setminus K(f_c)\}, \quad \boldsymbol{\Delta} = \{(c, z) \in \mathbb{C}^2 : z \in \Delta_c\};$$

these are open sets in $\mathbb{C}^2$. Let us also consider the critical set

$$\mathcal{C}_- = \{(c, z) \in \mathbf{F} : \exists n \geq 0, \ f_c^n(z) = 0\};$$

it is an analytic subvariety in $\mathbf{F}$. The Böttcher function

$$\mathbf{B} : \boldsymbol{\Delta} \to \mathbb{C} \setminus \overline{\mathbb{D}}, \quad (c, z) \mapsto B_c(z),$$

is a local holomorphic submersion, so that its level sets form a holomorphic foliation of $\boldsymbol{\Delta}$. Moreover, this foliation is transverse to the vertical foliation of $\mathbb{C}^2$, and thus determines a local holomorphic motion near any point $(c, z) \in \boldsymbol{\Delta}$.

Pulling this foliation back by the fiberwise dynamics $\mathbf{f} : (c, z) \mapsto (c, f_c z)$, we obtain a holomorphic foliation on $\mathbf{F}$ with singularities on $\mathcal{C}_-$. It determines a local holomorphic motion near any point $(c, z) \in \mathbf{F} \setminus \mathcal{C}_-$ that we call the *Böttcher motion*.

We say that some holomorphic motion over parameter domain *matches the Böttcher motion* or *respects the Böttcher coordinate* if on the basin of infinity it coincides with the Böttcher motion. Such a motion preserves the external angles and heights of the points in the basin of infinity.

3.2. Transversality to the diagonal

Let $\mathcal{C}_1 = \{(c, c) : c \in \mathbb{C} \setminus \mathcal{M}\}$.

Lemma 3.1. *Near any point $(c, c) \in \mathcal{C}_1$, the Böttcher motion is well defined and is transverse to $\mathcal{C}_1$.*

Proof. The Böttcher motion is well defined since $\mathcal{C}_1 \cap \mathcal{C}_- = \emptyset$. It is transverse to $\mathcal{C}_1$ since the Böttcher function $\mathbf{B}|\mathcal{C}_1$ is non-singular (as it conformally maps $\mathcal{C}_1$ onto $\mathbb{C} \setminus \mathcal{M}$). □

Let c_0 be a *Misiurewicz parameter*, i.e., there is a repelling periodic point a_0, of period q, such that $f_{c_0}^n(0) = a_0$ for some $n \geq 1$, assumed to be minimal with this property. There are finitely many (and at least two) dynamical rays $\mathcal{R}_{\theta_i}^{\mathrm{dyn}}(c_0)$ landing at c_0. Through a neighborhood of c_0, the Böttcher motion of these dynamical rays is well defined (see Lemma B.1 of [GM] and Lemma 2.2 of [Sc1]). Their common landing point $p(c)$ is just the analytic continuation of c_0 as a preperiodic point (that is, $p(c)$ is the solution of $f_c^{n-1}(z) = f_c^{n-1+q}(z)$ near c_0).

Lemma 3.2 (compare [vS]). *Let c_0 be a Misiurewicz parameter as above. Then the curve $c \mapsto p(c)$ is transverse to the diagonal $c \mapsto c$ at c_0.*

Proof. Let us consider one of the dynamical rays $\mathcal{R}_{\theta_i}^{\mathrm{dyn}}(c)$ landing at $p(c)$ which moves holomorphically under the Böttcher motion $h_c = B_c^{-1} \circ B_{c_0}$. By the basic dynamical-parameter relation, $\{c : c \in \mathcal{R}_{\theta_i}^{\mathrm{dyn}}(c)\}$ is a parameter ray $\mathcal{R}_{\theta_i}^{\mathrm{par}}$ landing at c. Moreover, the map $\gamma : c \mapsto h_c^{-1}(c)$ is a homeomorphism from $\mathcal{R}_{\theta_i}^{\mathrm{par}}$ to $\mathcal{R}_{\theta_i}^{\mathrm{dyn}}(c_0)$. But if the curves $c \mapsto p(c)$ and $c \mapsto c$ had tangency of order $d \geq 1$ at c_0 then each point on $\mathcal{R}_{\theta}^{\mathrm{dyn}}(c_0)$ would have $d+1$ preimages under γ (compare Lemma 9.1 of [ALM])—contradiction. $\square$

3.3. Parabolic wakes

Let $\mathcal{A}$ stand for the set of parameters c for which the map $f_c : z \mapsto z^d + c$ has an attracting fixed point α_c. In the quadratic case, it is a domain bounded by the main cardioid of the Mandelbrot set. In the higher degree case, $\mathcal{A}$ is a domain bounded by a Jordan curve with $d-1$ cusps.

The set $\mathcal{M} \setminus \overline{\mathcal{A}}$ is disconnected. The closures of the connected components of $\mathcal{M} \setminus \overline{\mathcal{A}}$ are called *(parabolic) limbs* of $\mathcal{M}$. Each limb $\mathcal{L}$ intersects $\overline{\mathcal{A}}$ at a single point called the *root* $r = r_{\mathcal{L}}$ of the limb. The map f_r has a parabolic fixed point with some multiplier $e^{2\pi i \mathbf{p}/\mathbf{q}}$. There are two parameter external rays landing at the root. Their union with r divides $\mathbb{C}$ into two (open) connected components: the one containing $\mathcal{L} \setminus \{r\}$ is called a *parabolic wake* $\mathcal{W} = \mathcal{W}_{\mathcal{L}}$ (see [DH1, M3, Sc1]).

For $c \in \mathcal{L}$, the map f_c has a unique *dividing* fixed point α_c. There are $\mathbf{q}$ external rays $\mathcal{R}_i^{\mathrm{dyn}}(c)$ landing at this point which are cyclically permuted by f_c with *combinatorial rotation number* $\mathbf{p}/\mathbf{q}$. This configuration of $\mathbf{q}$ rays, together with the α-fixed point, moves holomorphically over the whole parabolic wake $\mathcal{W}$. We let

$$\Gamma^0 = \Gamma^0(c) = \bigcup \mathcal{R}_i^{\mathrm{dyn}}(c).$$

Given some height $\xi > 0$, let $\mathcal{W}(\xi)$ stand for the domain obtained by truncating the parabolic wake $\mathcal{W}$ by the parameter equipotential $\mathcal{E}_{\xi}^{\mathrm{par}}$ of height ξ. For $c \in \mathcal{W}(d\xi)$, the *Yoccoz puzzle pieces* $Y_j^0 = Y_j^0(c)$ *of depth* 0 are obtained by taking the closure of the connected components of $\mathbb{C} \setminus (\Gamma^0(c) \cup \{\alpha_c\})$ truncated by the dynamical equipotential $\mathcal{E}_{\xi}^{\mathrm{dyn}}(c)$ of height ξ (where the piece containing 0 is also denoted Y^0). This configuration of $\mathbf{q}$ puzzle pieces moves holomorphically over $\mathcal{W}(d\xi)$.

Since $f_c(0) \notin \Gamma^0(c) \cup \mathcal{E}_{\xi}^{\mathrm{dyn}}(c)$ for $c \in \mathcal{W}(\xi)$, the f_c-preimages of the rays $\Gamma^0(c)$ move holomorphically over $\mathcal{W}(\xi)$, and so do $f_c^{-1}(\mathcal{E}_{\xi}^{\mathrm{dyn}}(c)) = \mathcal{E}_{\xi/d}^{\mathrm{dyn}}$ and the f_c-preimages of α_c. The closures of the components of $\mathbb{C} \setminus f_c^{-1}(\Gamma^0(c) \cup \alpha_c)$ truncated by the equipotential $\mathcal{E}_{\xi/d}^{\mathrm{dyn}}(c)$ are called *Yoccoz puzzle pieces of depth* 1, and are denoted Y_j^1 (where the one containing 0 is also denoted Y^1).

We now fix some height ξ (say, $\xi = 1$): the moduli bounds in what follows will depend on this choice, but it will not be explicitly indicated.

3.4. Satellite copies of $\mathcal{M}$

Let

$$\mathcal{M}_{\mathcal{L}} = \{c \in \mathcal{L} : f^{\mathbf{q}n}(0) \in Y^1, \, n = 0, 1, 2, \ldots\}.$$

This set is canonically homeomorphic to $\mathcal{M}$, and is called a *satellite copy* of the Multi-brot set (see [DH2, Sc2]). The maps f_c with $c \in \mathcal{M}_{\mathcal{L}} \setminus \{r_{\mathcal{L}}\}$ (and the corresponding parameters c) are called *satellite renormalizable*.

3.5. Decorations and Misiurewicz wakes

Removing the satellite copy from the limb $\mathcal{L}$ disconnects it into countably many components, each attached to $\mathcal{M}_{\mathcal{L}}$ at a Misiurewicz parameter c_* such that $f_{c_*}^{\mathbf{nq}}(0) \in f_{c_*}^{-1}(\alpha) \setminus \{\alpha\}$ for some $\mathbf{n} > 0$. The closures of these sets are called *decorations*.

There are $\mathbf{q}$ rays landing at c_*, dividing $\mathbb{C}$ into $\mathbf{q} - 1$ *Misiurewicz wakes* and the component containing $\mathcal{M}_{\mathcal{L}} \setminus \{c_*\}$. The above number $\mathbf{n}$ is called the *level* of the Misiurewicz wake and the corresponding decoration.

For c in the Misiurewicz wake, the level $\mathbf{n}$ is determined as the minimal natural number n such that $f_c^{n\mathbf{q}}(0)$ belongs to some $Y_j^1(c) \neq Y^1(c)$. Let $\mathcal{O}_k^{\mathbf{n}}$ stand for the Misiurewicz wakes truncated by the parameter equipotential of height $\xi/d^{\mathbf{qn}}$. Obviously, truncated Misiurewicz wakes are compactly contained in the corresponding truncated parabolic wake $\mathcal{W}(d\xi)$, and the Misiurewicz wakes attached to different roots have disjoint closures.

Define

$$\Omega^{\mathbf{n}} = \mathcal{W}(\xi/d^{\mathbf{nq}-1}) \setminus \bigcup_{\mathbf{m} < \mathbf{n}} \bigcup_k \overline{\mathcal{O}}_k^{\mathbf{m}}.$$

It is an open Jordan disk containing $\overline{\mathcal{O}}_k^{\mathbf{n}}$.

For the further understanding of the wakes, we need to go deeper into the puzzle.

The *Yoccoz puzzle pieces of depth n* are the pullbacks of Yoccoz puzzle pieces of depth 0 under f^n. The puzzle pieces of depth n will be denoted by Y_j^n, where the labels j stand for the angles of the external rays that bound Y_j^n. They form a tiling of the neighborhood of $K(f)$ bounded by the equipotential of height ξ/d^n.

We also let Y^n stand for the *critical* puzzle piece of depth n, i.e., $Y^n \ni 0$, while Y_v^n stands for the puzzle piece containing the critical *value*.

We call $\overline{\Omega}^{\mathbf{n}}$ the *parapuzzle piece of depth* $\mathbf{nq}$, containing $\mathcal{M}_{\mathcal{L}}$. The closure of the Misiurewicz wake $\overline{\mathcal{O}}_k^{\mathbf{n}}$ will also be called a parapuzzle piece of depth $\mathbf{nq} + 1$. We will now give a construction of the "parapuzzle pieces of depth $n \geq \mathbf{nq} + 2$" so that they will be the closures of the parameter domains over which the puzzle pieces of depth n move holomorphically (with the same "combinatorics"). Moreover, they will form a tiling of the Misiurewicz wake, appropriately truncated.

Lemma 3.3. *Fix some Misiurewicz wake $\mathcal{O} = \mathcal{O}_k^{\mathbf{n}} \subset \mathcal{W}$. Then all the boundaries of puzzle pieces up to depth $\mathbf{qn}$ move holomorphically over $\Omega^{\mathbf{n}}$, while the boundaries of puzzle pieces of depth $\mathbf{qn} + 1$ move holomorphically over $\mathcal{O}$. All these motions provide us*

with tubings over $\mathcal{O}$ respecting the Böttcher coordinate. The critical value $c \mapsto c = f_c(0)$ is a diagonal of the tubing of $\partial Y_v^{\mathbf{qn}}$ over $\mathcal{O}^{\mathbf{n}}$. Moreover, for any $c \in \mathcal{O}$,

$$\operatorname{mod}(Y^{\mathbf{nq}}(c) \setminus Y^{\mathbf{nq}+1}(c)) \geq \delta(\mathcal{O}) > 0. \tag{3.2}$$

Proof. Since for $c \in \Omega^{\mathbf{n}}$, the critical orbit $f_c^k(0)$, $k = 0, 1, \ldots, \mathbf{qn}$, does not cross the rays $\mathcal{R}_i^{\mathrm{dyn}}(c)$ and the equipotential $\mathcal{E}_\xi^{\mathrm{dyn}}(c)$, the configuration of Yoccoz puzzle pieces up to depth $\mathbf{qn}$ moves holomorphically over $\Omega^{\mathbf{n}} \ni \mathcal{O}$.

Similarly, for $c \in \mathcal{O}$, the critical value $f_c^{\mathbf{qn}+1}(0)$ does not cross the rays $\mathcal{R}_i^{\mathrm{dyn}}(c)$ and the equipotential $\mathcal{E}_\xi^{\mathrm{dyn}}(c)$ either, so that the puzzle pieces of depth $\mathbf{qn} + 1$ move holomorphically over $\mathcal{O}$. By Lemma 2.3, this motion is continuous up to the boundary of $\mathcal{O}$.

We see that the boundary of each puzzle piece up to depth $\mathbf{qn} + 1$ provides us with a tubing over $\mathcal{O}$. This tubing respects the Böttcher coordinate as it is induced by it.

Let us consider the puzzle piece $Y_v^{\mathbf{qn}}(c)$ moving holomorphically over $\Omega^{\mathbf{n}} \supset \mathcal{O}$ under the Böttcher motion h_c. It is bounded by two arcs of external rays with some angles θ_+ and θ_- (landing at the same point $a = a(c)$ such that $f^{\mathbf{qn}}a = \alpha$), and an arc of the equipotential $\mathcal{E}_{\xi/d^{\mathbf{qn}}}^{\mathrm{dyn}}$. By the basic relation (3.1), the Misiurewicz wake $\mathcal{O}$ is bounded by two arcs of external rays $\mathcal{R}_{\theta_\pm}^{\mathrm{par}}$ (landing at the Misiurewicz root c_* such that $f_{c_*}^{\mathbf{qn}}(c_*) = \alpha$) and an arc of the equipotential $\mathcal{E}_{\xi/d^{\mathbf{qn}}}^{\mathrm{par}}$. Moreover, the parameter-phase map

$$\gamma : \partial\mathcal{O} \to \partial Y_v^{\mathbf{qn}}, \qquad c \mapsto h_c^{-1}(c),$$

carries a parameter $c \in \partial\mathcal{O} \setminus \{c_*\}$ to the dynamical point $\gamma(c) \in Y_v^{\mathbf{qn}} \setminus \{a\}$ with the same Böttcher coordinates. This shows that the map $c \mapsto c$ satisfies properties (D1), (D2) and (D5) of the diagonal to the tubing of $\partial Y_v^{\mathbf{qn}}$ over $\mathcal{O}$. It satisfies (D3) and (D4) by the discussion in §3.2.

Since the holomorphic motion h over $\mathcal{O}$ extends to $\Omega^{\mathbf{n}}$,

$$\operatorname{mod}(Y^0(c) \setminus Y_j^1(c)) \geq \delta_{\mathbf{n}} > 0, \qquad c \in \mathcal{O}, \tag{3.3}$$

for any non-critical puzzle piece Y_j^1 contained in Y^0. Since

$$f^{\mathbf{qn}} : Y^{\mathbf{qn}}(c) \setminus Y^{\mathbf{qn}+1}(c) \to Y^0(c) \setminus Y_j^1(c), \qquad c \in \mathcal{O},$$

is a covering of degree d (for an appropriate non-critical puzzle piece Y_j^1), we obtain (3.2). $\qquad\square$

3.6. Puzzle motion over the parapuzzle

Let

$$\Gamma^n = \Gamma^n(c) = f^{-n}(\Gamma^0) \setminus f^{-(n-1)}(\Gamma^0).$$

Thus, Γ^n is the "new" ray boundary of the puzzle pieces of depth n (which is not contained in the ray boundary of the puzzle pieces of depth $n - 1$).

We say that f *has well defined combinatorics up to depth n* if 0 belongs to the interior of a puzzle piece of depth n. Note that for $c \in \mathcal{W}(\xi/d^{\mathbf{q}})$ with combinatorial rotation number $\mathbf{p}/\mathbf{q}$, combinatorics is well defined up to depth $\mathbf{q} + 1$.

There are three ways the combinatorics can fail to be well defined at level n:

- The Julia set $J(f)$ is connected and $f^n(0) = \alpha$. Such maps will be called α-*Misiure-wicz*.
- The Julia set is disconnected and the critical value $f^n(0)$ has height $\geq d\xi$. This situation will be essentially avoided by appropriate shrinking of the parameter domains.
- The Julia set $J(f_c)$ is disconnected and $f^n(0) \in \Gamma^0$. In this case, there are d rays in Γ^n that land at a precritical point. We call such precritical points *pinching points of depth n* (if n is the minimal integer with this property). Note that the pinching points of depth n belong to the interior of the puzzle piece of depth $n - 1$ (since by definition, they do not belong to $f^{-(n-1)}(\Gamma^0)$).

The *combinatorics of f up to depth n* (provided it is well defined) is the label of the puzzle piece of depth $n - 1$ containing the critical value.

As we saw in §3.5, all the maps f_c, $c \in \mathcal{O}$, have well defined combinatorics up to depth $\mathbf{qn}$ (and moreover, $\mathcal{O}$ is the maximal domain on which this is the case). We will now tile $\overline{\mathcal{O}}$ (truncated by appropriate equipotentials) according to the deeper combinatorics of the puzzle.

Lemma 3.4. *The set of parameters $c \in \mathcal{O}$ with the same combinatorics v up to depth $n \geq \mathbf{qn} + 1$ is an open Jordan disk bounded by the rays and equipotentials with the same angles and heights as the puzzle piece Y_v^{n-1} containing the critical value. The closure of this disk, $\mathcal{Y}_v^n$, is called the parapuzzle of depth n with combinatorics v. The boundaries of puzzle pieces of depth n provide us with Böttcher tubings over $\mathcal{Y}_v^n$ that fit the tubings of the boundaries of puzzle pieces of depth $< n$ containing it. The critical value $\psi : c \mapsto c = f_c(0)$, $c \in \mathcal{Y}_v^n$, is a diagonal to the tubing of ∂Y_v^{n-1}. The parapuzzle pieces of depth n tile the Misiurewicz wake $\mathcal{O}$ truncated by the equipotential of height ξ/d^{n-1}.*

Proof. Assume inductively that the statement is true up to depth n (where the base of induction is provided by the closed Misiurewicz wake $\overline{\mathcal{O}} = \mathcal{Y}^{\mathbf{qn}+1}$, see Lemma 3.3). Consider one of the parapuzzle pieces, $\mathcal{Y} = \mathcal{Y}_j^n \subset \overline{\mathcal{O}}$, and let us show how to tile its truncation by parapuzzle pieces of depth $n + 1$.

Let us consider the boundary of a puzzle piece $X = Y_k^n$ contained in Y_v^{n-1}. By the induction assumption, it provides us with a tubing over $\mathcal{Y}$ that fits the tubing of ∂Y_v^{n-1}, and ψ is a diagonal to the latter. Hence we can apply Lemma 2.1 and obtain an embedding $\chi : \partial X \to \mathcal{Y}$. The closed disk bounded by this Jordan curve is our parapuzzle $\mathcal{X}$ of depth $n + 1$. Moreover, the map ψ is the diagonal of the tubing of ∂X over $\mathcal{X}$. Properties (D1), (D2) and (D5) of the diagonal follow directly from the construction, while properties (D3) and (D4) follow from the discussion of §3.2.

Since ∂X moves under the Böttcher motion and the diagonal ψ is the identity $c \mapsto c$, the phase-parameter map $\chi : X \to \mathcal{X}$ respects the Böttcher coordinates. Hence the external angles and the heights of the rays and equipotentials forming $\partial \mathcal{X}$ are the same as those of ∂X.

Let us now consider the puzzle pieces Y_k^{n+1} of depth $n + 1$. Since $f_c(0) \notin \partial Y_j^n(c)$ for $c \in \mathcal{X}$ and any j, these puzzle pieces move holomorphically over int $\mathcal{X}$ (obviously, respecting the Böttcher coordinates). This motion is continuous up to the boundary by Lemma 2.3. Let us show that it fits the tubing of the boundary of the puzzle piece of depth n (and then inductively, of all smaller depths) containing it. Indeed, let $Y_l^{n+1} \subset Y_k^n$, and let $\zeta \in \partial Y_l^{n+1} \setminus \partial Y_k^n$. Then $\zeta \in \Gamma^{n+1}$, and since ∂Y_l^{n+1} moves under the Böttcher motion, $h_c(\zeta) \in \Gamma^{n+1}(c)$ for any $c \in \mathcal{X}$. Hence $h_c(\zeta) \notin f_c^{-n}(\Gamma^0(c))$, while the latter set contains $\partial Y_k^n(c)$. This provides us with the desired tubings of depth $n + 1$ over $\mathcal{X}$.

Finally, the puzzle pieces Y_k^n tile the puzzle piece Y_v^{n-1} truncated by the equipotential of height ξ/d^n and their simultaneous motion over $\mathcal{Y}$ fits the tubing of ∂Y_v^{n-1}. Applying Lemma 2.1 once again, we conclude that the corresponding parapuzzle pieces $\mathcal{X} = \mathcal{Y}_k^{n+1}$ tile the puzzle piece $\mathcal{Y}$ truncated by the equipotential of height ξ/d^n. $\qquad\square$

The parapuzzle piece of depth n containing a point c in its interior will also be denoted $\mathcal{Y}^n(c)$ (or $\mathcal{Y}^n$ when the choice of the base point c is self-evident or non-essential). For instance, for $c \in \mathcal{O} = \mathcal{O}_k^{\mathbf{n}}$, we have

$$\mathcal{Y}^{\mathbf{nq}} = \overline{\Omega^{\mathbf{n}}}, \qquad \mathcal{Y}^{1+\mathbf{nq}} = \overline{\mathcal{O}_k^{\mathbf{n}}}.$$

For $c \in \operatorname{int} \mathcal{Y}^n$, let $L_n = L_{n,c} : D_n \to Y^n$ be the *first landing map* to the critical puzzle piece Y^n. For $n \geq \mathbf{nq} + 1$, its domain $D_n = D_n(c)$ consists of disjoint puzzle pieces $W_i^n = W_i^n(c)$ each of which is univalently mapped by L_n onto Y^n. Note that $\mathbb{C} \setminus \operatorname{int} D_n$ is a forward invariant set, and int D_n contains a dense subset of the filled-in Julia set $K(f)$.

Lemma 3.5. *For $n \geq \mathbf{nq} + 1$, the set $\mathbb{C} \setminus \operatorname{int} D_n$ moves holomorphically over $\operatorname{int} \mathcal{Y}^n$. This motion is equivariant, i.e., $h_c \circ f_{c_0} = f_c \circ h_c$, and respects the Böttcher coordinate.*

Proof. We fix some base parameter $c_0 \in \mathcal{Y}^n$, and let $f = f_{c_0}$, $Y^n = Y^n(c_0)$, etc.

Let us first show that the boundary of each domain $W = W_i^n$ moves holomorphically over int $\mathcal{Y}^n$. Let $L_n | W = f^l$. Then W has an itinerary $(\mu_0, \ldots, \mu_{l-1}, \mu_l = 0)$, where $\mu_m \neq 0$ for $m < l$, satisfying the property

$$f^m(W) \subset Y_{\mu_m}^n, \qquad m = 0, \ldots, l.$$

For $c = c_0$, the restrictions $f | Y_{\mu_m}^n$ are univalent and

$$f(Y_{\mu_m}^n) \supset Y_{\mu_{m+1}}^n, \qquad m = 0, \ldots, l-1.$$

Since the puzzle pieces Y_μ^n move holomorphically over int $\mathcal{Y}^n$, the same property is valid for all $c \in \operatorname{int} \mathcal{Y}^n$. Now, the repeated application of Lemma 2.3 shows that the boundaries of all $f^m(W)$, $m = l-1, \ldots, 0$, move holomorphically over int $\mathcal{Y}^n$ as well.

By Lemma 3.4, for $c \in \operatorname{int} \mathcal{Y}^n$, $f_c(0) \in \operatorname{int} Y_v^{n-1}(c)$. Thus $0 \in \operatorname{int} Y^n(c) \subset \operatorname{int} D_n(c)$, and this implies that all precritical points are contained in int $D_n(c)$. Hence the Böttcher motion is well defined on $\mathbb{C} \setminus (\operatorname{int} D_n \cup K(f_c))$ (which is a dense subset of $\mathbb{C} \setminus \operatorname{int} D_n$). By

the λ-lemma, this Böttcher motion extends to the whole set $\mathbb{C} \setminus \text{int } D_n$, and this extension matches the previously constructed motion of $\bigcup_i \partial W_i^n$. The conclusion follows. □

We let $h_c^{(n)}$ be the motion of $\mathbb{C} \setminus \text{int } D_n$ over $\mathcal{Y}^n$ described in Lemma 3.5.

We say that a puzzle piece $Y = Y_\mu^k(c_0)$ *persists over depth n* if the boundary $\partial Y_\mu^k(c)$ moves holomorphically over $\text{int } \mathcal{Y}^n$ respecting the Böttcher coordinate. By Lemma 2.3, if $n \geq 1 + \mathbf{qn}$, this motion is continuous up to the boundary of $\mathcal{Y}^n$ and hence provides us with a tubing over $\mathcal{Y}^n = \mathcal{Y}^n(c_0)$. For instance, any puzzle piece Y_μ^n persists over depth n.

We say that a puzzle piece Y_μ^k is *subordinate to depth n* if it is not properly contained in any domain W_i^n.

Lemma 3.6. *Let $n \geq \mathbf{nq} + 1$. If a puzzle piece $Y = Y_\mu^k$ is subordinate to depth n, then it persists over depth n. Moreover, its motion fits the tubing of the boundary of any bigger puzzle piece $Z = Y_\nu^l$ over $\mathcal{Y}^n$.*

Proof. The first assertion follows from Lemma 3.5 since the boundary of Y_μ^k is contained in $\mathbb{C} \setminus \text{int } D_n$.

Let us verify the second assertion. Since "fitting" is a transitive property, it is sufficient to check it for two consecutive depths, $l = k - 1$. We may assume that $k > n$, since for $k \leq n$ the result follows from Lemma 3.4. Let us consider a puzzle piece $Y' = f^{k-n}(Y)$ of depth n, and let $Z' = f^{k-n}(Z)$. The latter is a puzzle piece of depth $n-1$ containing Y'.

Let h_c be the motion of $\mathbb{C} \setminus \text{int } D_n$ from Lemma 3.5. Since it is equivariant up to the boundary of $\mathcal{Y}^n$, we have

$$f_c^{k-n}(h_c(\partial Z)) = h_c(\partial Z'), \qquad c \in \mathcal{Y}^n. \tag{3.4}$$

By Lemma 3.4, $\partial Y'$ provides us with a tubing over $\mathcal{Y}^n$ that fits the tubing of $\partial Z'$. By (3.4), this property lifts to imply that the tubing of ∂Y fits the tubing of ∂Z. [Indeed, if $z \in \partial Y \cap \text{int } Z$ then $f_{c_0}(z) \in \partial Y' \cap \text{int } Z'$. Since the tubing of $\partial Y'$ fits the tubing of $\partial Z'$, $h_c(f_{c_0}^{k-n}(z)) = f_c^{k-n}(h_c z)$ does not belong to $h_c(\partial Z')$ for $c \in \mathcal{Y}^n$. By (3.4), $h_c z$ does not belong to $h_c(\partial Z)$.] □

A critical puzzle piece Y^n is called a *child* of a critical puzzle piece Y^m ($m < n$) if the map $f^{n-m} : Y^n \to Y^m$ is unicritical.

Corollary 3.7. *Assume that for some $n \geq \mathbf{nq}+1$ and $k \geq 1$, the map $f^k|Y^n$ is unicritical (e.g., Y^n is a child of some puzzle piece Y^m and $k \in [1, n - m]$). Then the motion $h^{(n)}$ provides us with a tubing of $\partial f^k(Y^n)$ over $\mathcal{Y}^n$, and the critical value $c \mapsto f_c^k(0)$ is a diagonal to this tubing.*

Proof. The first assertion follows from Lemma 3.6 since the piece $f^k(Y^n)$ is subordinate to depth n. The second assertion follows from Lemma 3.4 for $k = 1$. Applying the family of univalent maps $f_c^{k-1} : f_c(Y^n(c)) \to f^k(Y^n(c))$, we obtain it for any k. □

If f and $\tilde{f}$ have the same combinatorics up to depth n, a *(Böttcher marked) pseudo-conjugacy* (up to depth n) between f and $\tilde{f}$ is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $H \circ f = \tilde{f} \circ H$ everywhere outside $\text{int } Y^n$, and which is the identity near infinity with respect to the Böttcher coordinates.

Remark 3.1. By Lemmas 4.2 and 4.3 of [AKLS], if c and $\tilde{c}$ have the same combinatorics up to depth n, and there exists a K-qc homeomorphism $(\text{int } Y^n(c), 0) \to (\text{int } Y^n(\tilde{c}), 0)$ which is the identity on the boundary with respect to the Böttcher coordinates, then f_c and $f_{\tilde{c}}$ are K-qc pseudo-conjugate (up to depth n).

3.7. Combinatorics of children

If f does not have well defined combinatorics of all depths, then either the Julia set of f is disconnected or the critical point is eventually mapped to the repelling fixed point α. Otherwise, we have critical puzzle pieces of all depths. In this case, we say that f is *combinatorially recurrent* if the critical point returns to all critical puzzle pieces.

Given a critical puzzle piece Y^n, let R_{Y^n} be the first return map to Y^n. The components of the domain of R_{Y^n} are puzzle pieces, which are mapped by R_{Y^n} onto Y^n, either univalently (if the component is non-critical), or d-to-1 (if the component is critical). Let $\mathrm{m}(Y^n)$ be the infimum, over all components D of the domain of R_{Y^n}, of $\mathrm{mod}(Y^n \setminus D)$.

If f is combinatorially recurrent, then every critical puzzle piece has a child. These kids are ordered by "age": a child Y^k is "older" than a child Y^l if $Y^k \supset Y^l$ (and thus $k \leq l$). Note that the *first child* Y^k of Y^n coincides with the critical component of the domain of R_{Y^n}.

A combinatorially recurrent map is said to be *primitively renormalizable* if there exists a critical puzzle piece Y^n such that the critical point never escapes its first child Y^k under iterates of R_{Y^n}: $\{R_{Y^n}^j(0) : j \geq 1\} \subset Y^k$. In general, we will say that a map f is *non-renormalizable* if it is neither satellite nor primitively renormalizable.

A child $Q = Y^q$ of $V = Y^v$ is called *good* if $f^{q-v}(0)$ is contained in the first child $U = Y^u$ of V. In this situation, $K = Y^{q-v+u}$ is a child of U called the *friend* of Q. Note that $f^{q-v} : Q \setminus K \to V \setminus U$ is a covering map of degree d.

The *favorite child* of V is the oldest good child Q that appears after the first child U. One can see that the depth of the favorite child is the smallest integer $q > v$ such that $f^{q-v}(0)$ belongs to the first child U and the orbit $\{f^i(0)\}_{i=1}^{q-v}$ passes through the annulus $V \setminus U$ (see the discussion preceding Lemma 2.3 of [AKLS]). If f is combinatorially recurrent and non-renormalizable, then every critical puzzle piece has a favorite child.

3.8. Phase-parameter transfer

We will now apply Lemmas 2.1 and 2.2 to two dynamical situations that will often appear in what follows.

Lemma 3.8. *Let us consider four levels $s < t \leq u < w$, where $u \geq \mathbf{nq} + 1$, such that Y^u is a child of Y^s and $f^{u-s}(Y^w)$ is contained in some connected component $W = W_i^t$ of the first landing map to Y^t. Assume that*

(1) $\mathrm{mod}(Y^s(c) \setminus W(c)) > \delta$ *for any $c \in \mathcal{Y}^u$;*
(2) f_c *and* $f_{\tilde{c}}$ *are K-qc pseudo-conjugate up to depth t for any $c, \tilde{c} \in \mathcal{Y}^u$.*

Then $\mathrm{mod}(\mathcal{Y}^u \setminus \mathcal{Y}^w) > \epsilon(K, \delta)$.

Proof. Notice that Y^s and W are subordinate to depth u, and $c \mapsto f_c^{u-s}(0)$ is a diagonal to the tubing of ∂Y^s over $\mathcal{Y}^u$. The first assumption implies that $\mathrm{diam}_{Y^s(c)} W(c) < M = M(\delta)$ and the second assumption implies that the maps $h_c^{(t)}|(\partial Y^s \cup \partial W)$ have K-qc extensions to the whole complex plane, $c \in \mathcal{Y}^u$. Application of the second statement of Lemma 2.2 gives the result:

$$\mathrm{mod}(\mathcal{Y}^u \setminus \mathcal{Y}^w) \geq \mathrm{mod}(\mathcal{Y}^u \setminus \mathcal{Y}^{\mathrm{depth}(W)+u-s}) > \epsilon(K, \delta). \qquad \square$$

Lemma 3.9. *Let us consider four puzzle pieces $K \subset Q \subset U \Subset V$ of respective depths $k > q \geq u > v \geq \mathbf{nq} + 1$. Assume that U is the first child of V, Q is a good child of V, and K is the friend of Q. Let $\mathcal{K} \subset \mathcal{Q} \subset \mathcal{U} \Subset \mathcal{V}$ be the corresponding parapuzzle pieces. Then*

$$\mathrm{mod}(\mathcal{Q} \setminus \mathcal{K}) > \rho(\mathrm{mod}(\mathcal{V} \setminus \mathcal{Q}))\, \mathrm{mod}(V \setminus U),$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function.

Proof. Let $A = V \setminus U$. Since this annulus persists over $\mathcal{U}$, all the maps

$$f_c : A \to f(A), \quad c \in \mathcal{U},$$

are coverings of degree d.

By Corollary 3.7, the puzzle piece $f(U)$ (and of course, $f(V)$) persists over $\mathcal{V}$, so that, the boundary of the annulus $f(A)$ moves holomorphically under $h_c = h_c^v$. Let us extend this motion to the whole annulus $f(A)$ (using the same notation for the extension). By Lemma 2.3, this motion lifts to a holomorphic motion H_c of A over $\mathrm{int}\,\mathcal{U}$ continuous up to the boundary. For any $z \in A \setminus \partial V$ and $c \in \mathcal{U}$, $h_c(f(z)) \notin \partial f(V)$, hence $H_c(z) \notin \partial V$. Thus, the motion of A fits the tubing of ∂V over $\mathcal{U}$.

Let us now consider a unicritical family $f_c^{q-v} : Q(c) \to V(c)$ over $\mathcal{Q}$ and a unicritical family $f_c^{q-v} : K(c) \to U(c)$ over $\mathcal{K}$. By Corollary 3.7, the critical value $c \mapsto f_c^{q-v}(0)$ is a diagonal to the corresponding tubings: of ∂V over $\mathcal{Q}$ and of ∂U over $\mathcal{K}$. Hence the corresponding phase-parameter map $\chi : V \to \mathcal{Q}$ maps the annulus A onto the annulus $\mathcal{Q} \setminus \mathcal{K}$. By Lemma 2.1, the dilatation of this map is bounded by the dilatation of the motion H_c over $\mathcal{Q}$, which is equal to the dilatation of $h_c|f(A)$ over $\mathcal{Q}$. By the λ-lemma, the latter is bounded by $\rho(\mathrm{mod}(\mathcal{V} \setminus \mathcal{Q}))$, which implies the desired estimate. $\qquad \square$

4. The favorite nest and the principal nest

4.1. The favorite nest

Let $Q^0 = Y^{\mathbf{nq}}$. Let Q^{i+1} be the favorite child of Q^i, and let P^i be the first child of Q^i. Let q_n and p_n be the depths of these puzzle pieces, i.e., $Q^n = Y^{q_n}$, $P^n = Y^{p_n}$. Let $k_n = q_n + p_{n-1} - q_{n-1}$ and $K^n = Y^{k_n}$. Note that

$$f^{q_n - q_{n-1}} : (Q^n, K^n) \to (Q^{n-1}, P^{n-1}).$$

By Proposition 2.4 of [AKLS], we have

$$\mathrm{mod}(Q^n \setminus P^n) > \delta(\mathcal{O}), \quad n \geq 0. \tag{4.1}$$

This implies in particular that

$$\text{mod}(Q^n \setminus K^n) = \frac{1}{d} \text{mod}(Q^{n-1} \setminus P^{n-1}) > \frac{1}{d}\delta(\mathcal{O}), \quad n \geq 1. \qquad (4.2)$$

Let us also consider the corresponding parapuzzle pieces: $\mathcal{Q}^n = \mathcal{Y}^{q_n}$, $\mathcal{P}^n = \mathcal{Y}^{p_n}$, and $\mathcal{K}^n = \mathcal{Y}^{k_n}$.

Theorem 4.1. *There exists $\delta > 0$, depending only on $\mathcal{O}$, such that for $n \geq 2$, $\text{mod}(K^n \setminus P^n) > \delta$, $\text{mod}(\mathcal{K}^n \setminus \mathcal{P}^n) > \delta$, and for $n \geq 3$, $\text{mod}(\mathcal{Q}^n \setminus \mathcal{K}^n) > \delta$.*

Proof. We start with the first and second estimates. The map $f^{q_n - q_{n-1}}$ is unicritical on Q^n; all the more, it is unicritical on P^n. It follows that $p_n - q_n \geq q_n - q_{n-1}$, and hence the puzzle piece

$$D = f^{q_n - q_{n-1}}(P^n) \subset P^{n-1}$$

is a component of the first landing map to Q^n. It follows that $f^{p_{n-1} - q_{n-1}}(D)$ is contained in a component of the first return map to Q^{n-1}. In particular

$$\text{mod}(K^n \setminus P^n) = \frac{1}{d} \text{mod}(P^{n-1} \setminus D) \geq \frac{1}{d^2} \text{m}(Q^{n-1})$$
$$\geq \frac{1}{d^3} \text{mod}(Q^{n-2} \setminus P^{n-2}) \geq \frac{1}{d^3}\delta(\mathcal{O}), \qquad (4.3)$$

where the last estimate follows from (4.1), while the previous one follows from Lemma 2.2 of [AKLS]. This proves the first estimate of the lemma.

For the second estimate, let us define $s < t < u < w$ as follows: $s = p_{n-1}$, $t = q_n$, $u = k_n$, $w = p_n$. Then (4.3) implies that condition (1) of Lemma 3.8 is satisfied. Condition (2) of Lemma 3.8 is satisfied by Theorem 4.4 (and Remark 4.1) of [AKLS]. Applying Lemma 3.8, we get the conclusion.

For the third estimate, let $s < t < u < w$ be as follows: $s = q_{n-1}$, $t = k_{n-1}$, $u = q_n$, $w = k_n$. Then $f^{u-s}(Y^w) = f^{q_n - q_{n-1}}(K^n) = P^{n-1}$ is contained in K^{n-1}, which is a (trivial) component of the first landing map to K^{n-1}. By (4.2), condition (1) of Lemma 3.8 is satisfied.

Furthermore, $\text{mod}(\mathcal{K}^{n-1} \setminus \mathcal{Q}^n) \geq \text{mod}(\mathcal{K}^{n-1} \setminus \mathcal{P}^{n-1}) \geq \delta$ by the previous estimate. By Remark 3.1 and the λ-lemma, this implies that condition (2) of Lemma 3.8 is satisfied with $K = K(\mathcal{O})$. The conclusion now follows from Lemma 3.8. $\square$

4.2. The principal nest

Let $V^0 = Y^{\mathbf{nq}}$. The *principal nest* starting at V^0 is the nest $V^0 \supset V^1 \supset \cdots$ obtained by taking V^{i+1} as the first child of V^i. Let v_i be the depth of V^i, i.e., $V^i = Y^{v_i}$. Let $\mathcal{V}^i = \mathcal{Y}^{v_i}$ stand for the corresponding parapuzzle pieces. By Corollary 3.7, ∂V^i provides us with a tubing over $\mathcal{V}^{i+1}$ with diagonal $c \mapsto f_c^{v_{i+1} - v_i}(0)$.

Let $g_i = g_{i,c} = R_{V^i}$. If the critical point returns to V^{i+1}, we let $s_i = s_i(c)$ be the smallest $k \geq 0$ such that $g_i^{k+1}(0) \in V^{i+1}$. (In other words, $s_i + 1$ is the first return time of

the critical point back to V^{i+1} under the iterates of g_i.) If $s_i = 0$ (that is, $g_i(0) \in V^{i+1}$), we say that the return to V^i is *central*.

The map $g_{i+1} = f^{v_{i+2}-v_{i+1}} : V^{i+2} \to V^{i+1}$ admits a unicritical "Koebe extension"

$$\mathbf{g}_{i+1} = \mathbf{g}_{i+1,c} : \mathbf{V}^{i+2} \to V^i, \quad \text{where} \quad \mathbf{V}^{i+2} = Y^{v_{i+2}-v_{i+1}+v_i} \equiv Y^{\mathbf{v}_{i+2}} \subset V^{i+1}.$$

(Note that $\mathbf{V}^{i+2}$ is a good child of V^i.) By Corollary 3.7, ∂V^i provides us with a tubing over the parapuzzle $\mathcal{V}^{i+2} \equiv Y^{\mathbf{v}_{i+2}}$ with diagonal $c \mapsto f_c^{v_{i+2}-v_{i+1}}(0)$.

Lemma 4.2. *We have the estimates*

$$\mathrm{mod}(V^{i+1} \setminus V^{i+2}) \geq \frac{1}{d}(\mathrm{mod}(V^i \setminus V^{i+1}) + s_i \mathrm{m}(V^i)), \tag{4.4}$$

$$\mathrm{mod}(\mathcal{V}^{i+1} \setminus \mathcal{V}^{i+2}) \geq \mathrm{mod}(\mathcal{V}^{i+2} \setminus \mathcal{V}^{i+2}) > \rho(\mathrm{mod}(\mathcal{V}^i \setminus \mathcal{V}^{i+1})) \mathrm{mod}(V^i \setminus V^{i+1}). \tag{4.5}$$

Proof. Let $D^j = Y^{d_j}$, $1 \leq j \leq s_i + 1$, be the pullback of V^i under g_i^j, so that

$$V^{i+1} = D^1 \supset \cdots \supset D^{s_i+1} = \mathbf{V}^{i+2}. \tag{4.6}$$

Notice that the D^j are all children of V^i.

For $1 \leq j \leq s_i$, $f^{d_j-v_i}(D^{j+1})$ is a non-central component of Dom g_i. It follows that $\mathrm{mod}(V^i \setminus f^{d_j-v_i}(D^{j+1})) \geq \mathrm{m}(V^i)$. Since

$$f^{d_j-v_i} : D^j \setminus D^{j+1} \to V^i \setminus f^{d_j-v_i}(D^{j+1})$$

is a covering of degree d, we have

$$\mathrm{mod}(D^j \setminus D^{j+1}) \geq \frac{1}{d}\mathrm{m}(V^i), \quad 1 \leq j \leq s_i. \tag{4.7}$$

Moreover, $f^{v_{i+2}-v_{i+1}} : \mathbf{V}^{i+2} \setminus V^{i+2} \to V^i \setminus V^{i+1}$ is a covering of degree d as well. Hence

$$\mathrm{mod}(\mathbf{V}^{i+2} \setminus V^{i+2}) = \frac{1}{d}\mathrm{mod}(V^i \setminus V^{i+1}). \tag{4.8}$$

Putting (4.7), (4.8) together with the Grötzsch inequality, we get (4.4).

Applying Lemma 3.9 to the nest $V^{i+2} \subset \mathbf{V}^{i+2} \subset V^{i+1} \subset V^i$, we obtain

$$\mathrm{mod}(\mathcal{V}^{i+2} \setminus \mathcal{V}^{i+2}) > \rho(\mathrm{mod}(\mathcal{V}^i \setminus \mathcal{V}^{i+2})) \mathrm{mod}(V^i \setminus V^{i+1}).$$

Since $\mathcal{V}^{i+1} \supset \mathcal{V}^{i+2}$, (4.5) follows. $\qquad\square$

Define $0 = i_0 < i_1 < \cdots$ to be the sequence of levels such that for $j > 0$ the return to $i_j - 1$ is non-central, i.e., $g_{i_j-1}(0) \notin V^{i_j}$.

Lemma 4.3. *Let $i_{j-1} \leq s < i_j$. Then we have the estimates*

$$\mathrm{mod}(V^{i_j} \setminus V^{i_j+1}) \geq \frac{1}{d}\mathrm{mod}(V^s \setminus V^{s+1}),$$

$$\mathrm{mod}(\mathcal{V}^{i_j} \setminus \mathcal{V}^{i_j+1}) > \rho(\mathrm{mod}(\mathcal{V}^s \setminus \mathcal{V}^{s+1})) \mathrm{mod}(V^s \setminus V^{s+1}).$$

Proof. Let n be the first moment such that $f^n(0) \in V^s \setminus V^{s+1}$, and let $m > 0$ be the first moment such that $f^{n+m}(0) \in V^{s+1}$. Then $Q = Y^{v_s+n+m}$ is the favorite child of V^s.

We have $Q \supset V^{i_j+1}$ since $V^s \supset V^{i_j}$ and $n + m$ is not great than the first return time to V^{i_j}. A similar argument gives $f^{n+m}(V^{i_j+1}) \subset V^{s+1}$. Hence

$$\mathrm{mod}(Q \setminus V^{i_j+1}) \geq \frac{1}{d} \mathrm{mod}(V^s \setminus V^{s+1}).$$

Since $Q \subset Y^{v_s+n} = V^{i_j}$, the first statement follows.

The second statement follows from Lemma 3.9 applied to the nest

$$K \subset Q \subset V^{s+1} \subset V^s,$$

where K is the friend of Q (since $f^{n+m}(V^{i_j+1}) \subset V^{s+1}$ implies $V^{i_j+1} \subset K$). $\square$

Lemma 4.4. *If $V^{i_j-1} \subset Q^n$ and V^{i_j+1} is defined then $V^{i_j+1} \subset Q^{n+1}$.*

Proof. Recall that $q_{n+1} = q_n + m$ where m is minimal with $f^m(0) \in P^n$ and $\{f^k(0) : 1 \leq k \leq m\} \cap Q^n \setminus P^n \neq \emptyset$. Clearly $V^{i_j} \subset P^n$, so we just have to show that $\{f^k(0) : 1 \leq k \leq v_{i_j+1} - v_{i_j}\} \cap Q^n \setminus P^n \neq \emptyset$, as this implies $q_{n+1} = q_n+m \leq v_{i_j}+m \leq v_{i_j}+v_{i_j+1}-v_{i_j}$.

Let $k \geq 1$ be minimal such that $f^k(0) \in Q^n \setminus P^n$. Then $k = l_0(p_n - q_n)$ for some $l_0 > 0$, and $R_{Q^n}^l(0) \in Y^{q_n+(l_0-l)(p_n-q_n)} \setminus Y^{p_n+(l_0-l)(p_n-q_n)}$, $1 \leq l \leq l_0$.

Since $R_{V^{i_j-1}}(0) \notin V^{i_j}$, we also have $R_{Q^n}(0) \notin V^{i_j}$, so that $V^{i_j} \subset Y^{q_n+(l_0-1)(p_n-q_n)}$. This clearly implies that $R_{Q^n}^l(0) \notin V^{i_j}$, $1 \leq l \leq l_0$, so $v_{i_j+1} - v_{i_j} \geq k$, as desired. $\square$

Theorem 4.5. *There exists $\delta > 0$, depending only on $\mathcal{O}$, such that*

(1) $\mathrm{mod}(V^{i_j} \setminus V^{i_j+1}) > \delta$,
(2) $\mathrm{mod}(\mathcal{V}^{i_j} \setminus \mathcal{V}^{i_j+1}) > \delta$.

Proof. By Lemma 3.3, the first statement holds for $j = 0$. Since $\mathcal{V}^0 = \mathcal{Y}^{\mathbf{nq}} = \Omega^{\mathbf{n}}$ and $\mathcal{V}^1 \subset \mathcal{Y}^{\mathbf{nq}+1} = \mathcal{O}$, we have $\mathrm{mod}(\mathcal{V}^0 \setminus \mathcal{V}^1) \geq \mathrm{mod}(\Omega^{\mathbf{n}} \setminus \mathcal{O}) = \delta(\mathcal{O})$, so the second statement also holds for $j = 0$.

By Lemmas 4.4 and 4.3, it is enough to show that for every $n \geq 3$ such that Q^{n+1} is defined, if $k \geq 0$ is maximal such that $V^k \supset Q^n$ then

$$\mathrm{mod}(V^{k+1} \setminus V^{k+2}), \mathrm{mod}(\mathcal{V}^{k+1} \setminus \mathcal{V}^{k+2}) > \delta.$$

Note first that since P^n and V^{k+2} are the first children of Q^n and V^{k+1} respectively, we have $V^{k+2} \subset P^n$. Recall the definition of K^n given at the beginning of Section 4.1: $K^n = Y^{q_n+p_{n-1}-q_{n-1}}$. If $V^{k+1} \supset K^n$, then

$$\mathrm{mod}(V^{k+1} \setminus V^{k+2}) \geq \mathrm{mod}(K^n \setminus P^n) \quad \text{and} \quad \mathrm{mod}(\mathcal{V}^{k+1} \setminus \mathcal{V}^{k+2}) \geq \mathrm{mod}(\mathcal{K}^n \setminus \mathcal{P}^n).$$

If $K^n \supset V^{k+1}$, then $\mathrm{mod}(V^k \setminus V^{k+1}) \geq \mathrm{mod}(Q^n \setminus K^n)$ and $\mathrm{mod}(\mathcal{V}^k \setminus \mathcal{V}^{k+1}) \geq \mathrm{mod}(\mathcal{Q}^n \setminus \mathcal{K}^n)$, so by Lemma 4.2,

$$\mathrm{mod}(V^{k+1} \setminus V^{k+2}) \geq \frac{1}{d} \mathrm{mod}(Q^n \setminus K^n),$$

$$\mathrm{mod}(\mathcal{V}^{k+1} \setminus \mathcal{V}^{k+2}) > \rho(\mathrm{mod}(\mathcal{Q}^n \setminus \mathcal{K}^n)) \, \mathrm{mod}(Q^n \setminus K^n).$$

In either case, the result follows from Theorem 4.1. $\square$

Remark 4.1. In [L3], [L4], it is shown that if $d = 2$ then one has better estimates

(1) $\mathrm{mod}(V^{i_j} \setminus V^{i_j+1}) > (j+1)\delta$,
(2) $\mathrm{mod}(\mathcal{V}^{i_j} \setminus \mathcal{V}^{i_j+1}) > (j+1)\delta$.

Remark 4.2 (Beau bounds). It follows from the arguments in [KL1], [KL2], [AKLS], and this work that there exists $\delta > 0$ (depending on the degree, but not on $\mathcal{O}$) such that for every j sufficiently large (depending on $\mathcal{O}$) one has

(1) $\mathrm{mod}(V^{i_j} \setminus V^{i_j+1}) > \delta$,
(2) $\mathrm{mod}(\mathcal{V}^{i_j} \setminus \mathcal{V}^{i_j+1}) > \delta$.

5. Slow recurrence

If $c \in \mathcal{M}$ is not combinatorially recurrent then either c has a non-repelling fixed point, or f_c is satellite renormalizable, or f_c is *semi-hyperbolic* (that is, its critical point is non-recurrent and belongs to the Julia set). It is well known that the set of semi-hyperbolic parameters has zero Lebesgue measure. Indeed, in [RL] a more precise version of the following is proved:

Theorem 5.1. *If $c \in \mathcal{M}$ is a semi-hyperbolic parameter then c is a Lebesgue density point of the complement of $\mathcal{M}$.*

In particular, almost every parameter in some $\mathcal{V}^{n+1}$ is either in the complement of $\mathcal{M}$ or is combinatorially recurrent. For real parameters, the corresponding statement has been proved in [Sa]: the set of semi-hyperbolic parameters $c \in \mathcal{M} \cap \mathbb{R}$ has zero one-dimensional Lebesgue measure. We will now concentrate on the analysis of combinatorially recurrent parameters.

Remark 5.1. A proof that the set of semi-hyperbolic parameters has zero Lebesgue measure can also be obtained along the lines of the approach we follow for combinatorially recurrent parameters given below.

5.1. Positive probability of slow recurrence

Lemma 5.2. *For any $\delta > 0$ and $\lambda > 0$ there exist $\epsilon = \epsilon(d, \delta)$ and $K = K(d, \delta, \lambda)$ with the following property. Assume that for some parapuzzle piece $\mathcal{V}^n$,*

$$\mathrm{mod}(\mathcal{V}^n \setminus \mathcal{V}^{n+1}) > \delta \quad and \quad \mathrm{mod}(V^n(c) \setminus V^{n+1}(c)) > \delta, \quad c \in \mathcal{V}^{n+1}.$$

Then for every $c \in \mathcal{V}^{n+k+3}$ such that $s_{n+i} \geq (1+i)K$, $i = 0, 1, \ldots, k$, we have:

(1) $\mathrm{mod}(V^{n+i} \setminus V^{n+i+1}) > \max\{\epsilon, (i-1)\lambda\}$,
(2) $\mathrm{mod}(\mathcal{V}^{n+i} \setminus \mathcal{V}^{n+i+1}) > \max\{\epsilon, (i-2)\lambda\}$,
(3) $\mathrm{mod}(\mathcal{V}^{n+i+2} \setminus \mathcal{V}^{n+i+2}) > \max\{\epsilon, (i-1)\lambda\}$,

for $i = 0, 1, \ldots, k+1$.

Proof. The inequalities $\mod(V^{n+i+1} \setminus V^{n+i+2}) \geq \frac{1}{d}(1 + s_{n+i})m(V^{n+i})$, given by (4.4), and $m(V^{n+i+1}) \geq \frac{1}{d} \mod(V^{n+i} \setminus V^{n+i+1})$, given by Lemma 2.2 of [AKLS], imply the estimate $\mod(V^{n+i} \setminus V^{n+i+1}) > \delta_i(1 + i)$, where $\delta_0 = \delta$, $\delta_1 = \delta/d$, and $\inf_{i \geq 1} \delta_i$ goes to infinity with K (given δ and d fixed). The first estimate follows.

Together with (4.5), it implies the rest.   □

Given a sequence of disjoint sets X^n, $n \geq 0$, on a probability space $\mathcal{V}$, we let $\mathbb{X}^n = \bigcup_{k<n} X^n$, $\mathbb{X} \equiv \bigcup_{k \geq 0} X^k$. Below we will make use of the following general formula:

$$1 - \mathbb{P}(\mathbb{X}) = \prod_{n \geq 0}(1 - \mathbb{P}(X^n \mid \mathcal{V} \setminus \mathbb{X}^n)) \tag{5.1}$$

(where $\mathbb{P}$ stands for probability or conditional probability). Indeed, letting $A^n = \mathcal{V} \setminus \mathbb{X}^n$, $A \equiv \bigcap_{n \geq 0} A^n = \mathcal{V} \setminus X$, we have $A^0 \supset A^1 \supset \cdots \supset A$, and

$$\mathbb{P}(A) = \prod_{n \geq 0} \mathbb{P}(A^{n+1} \mid A^n) = \prod_{n \geq 0} \mathbb{P}(\mathcal{V} \setminus X^n \mid \mathcal{V} \setminus \mathbb{X}^n),$$

which yields (5.1)

Lemma 5.3. *Assume that for some parameter c_0,*

$$\mod(\mathcal{V}^n \setminus \mathcal{V}^{n+1}) > \delta, \quad \mod(V^n \setminus V^{n+1}) > \delta.$$

Let $Z_{\mathrm{nr}} \subset \mathcal{V}^{n+1}$ be the set of parameters which are not combinatorially recurrent. Fix some $K > 0$ as in Lemma 5.2, and let $Z_{\mathrm{sr}} \subset \mathcal{V}^{n+1}$ be the set of combinatorially recurrent parameters for which $s_{n+k} \geq (1 + k)K$, $k \geq 0$.[6] Then $\mathbb{P}(Z_{\mathrm{sr}} \cup Z_{\mathrm{nr}} \mid \mathcal{V}^{n+1}) > \epsilon(\delta, d, K) > 0$.

Proof. We can assume that K is larger than the $K(\delta, d)$ given by Lemma 5.2. Let $t_k = (1 + k)K$, and let $\mathbb{X} = \mathcal{V}^{n+1} \setminus (Z_{\mathrm{sr}} \cup Z_{\mathrm{nr}})$. For $k \geq 0$, $0 \leq j < t_k$, let $X^{k,j} \subset \mathbb{X}$ be the set of all $c \in \mathbb{X}$ such that $s_{n+i} \geq t_i$, $0 \leq i < k$ and $s_{n+k} = j$. Notice that $\mathbb{X} = \bigsqcup_{(k,j)} X^{k,j}$. We order the pairs (k, j) lexicographically. As above, let

$$\mathbb{X}^{k,j} = \bigcup_{(k',j')<(k,j)} X^{k',j'}.$$

Notice that for $c \in X^{k,j}$, we have

$$\mathcal{V}^{n+k+2}(c) \cap \mathbb{X}^{k,j} = \emptyset \tag{5.2}$$

while

$$\mathcal{V}^{n+k+2}(c) \cap X^{k,j} \subset \mathcal{V}^{n+k+2}(c). \tag{5.3}$$

Indeed, for $\tilde{c} \in \mathcal{V}^{n+k+2}(c)$, we have $s_{n+i}(\tilde{c}) = s_{n+i}(c) \geq t_{n+i}$ for $i < k$, while $s_{n+k}(\tilde{c}) \geq s_{n+k}(c) = j$, with equality attained iff $\tilde{c} \in \mathcal{V}^{n+k+2}(c)$.

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[6] Label "sr" stands for "slow recurrent".

Together with (5.1), (5.2) gives us

$$\mathbb{P}(Z_{\mathrm{nr}} \cup Z_{\mathrm{sr}} \mid \mathcal{V}^{n+1}) = 1 - \mathbb{P}(\mathbb{X} \mid \mathcal{V}^{n+1}) = \prod_{(k,j)} (1 - \mathbb{P}(X^{k,j} \mid \mathcal{V}^{n+1} \setminus \mathbb{X}^{k,j}))$$

$$\geq \prod_{(k,j)} (1 - \sup_{c \in X^{k,j}} \mathbb{P}(X^{k,j} \mid \mathcal{V}^{n+k+2}(c)).$$

It is thus enough to prove an estimate such as

$$\mathbb{P}(X^{k,j} \mid \mathcal{V}^{n+k+2}(c)) \leq e^{-(1+k)\epsilon}, \qquad c \in X^{k,j},$$

for some $\epsilon = \epsilon(\delta, d)$. But this follows from (5.3) and the estimate

$$\mathrm{mod}(\mathcal{V}^{n+k+2}(c) \setminus \mathcal{V}^{n+k+2}(c)) \geq (1+k)\epsilon$$

of the previous lemma.   □

Remark 5.2. The above proof can be easily refined as follows. One can define Z_{sr} as the set of combinatorially recurrent parameters $c \in \mathcal{V}^{n+1}$ for which the sequence s_{n+i} satisfies $s_n, s_{n+1}, s_{n+2} \geq K$ and $s_{n+i+1} \geq 2^{s_{n_i}}$ for $i \geq 2$ (thus displaying "torrential growth" in the terminology of [AM1]). We would still obtain $\mathbb{P}(Z_{\mathrm{nr}} \cup Z_{\mathrm{sr}} \mid \mathcal{V}^{n+1}) > \epsilon$.

Let $\mathcal{S} \subset \mathcal{M}$ be the set of combinatorially recurrent parameters c such that s_n, $\mathrm{mod}(\mathcal{V}^n(c) \setminus \mathcal{V}^{n+1}(c))$ and $\mathrm{mod}(V^n(c) \setminus V^{n+1}(c))$ grow at least linearly with n.

Let Z_{r} be the set of combinatorially recurrent non-renormalizable parameters in $\mathcal{M}$.

Corollary 5.4. *For $c \in Z_{\mathrm{r}}$, there exist parapuzzle pieces $\mathcal{V}^{n+1}(c)$ of arbitrarily small diameter such that $\mathbb{P}(\mathcal{M} \setminus \mathcal{S} \mid \mathcal{V}^{n+1}(c)) < 1 - \delta$, with $\delta = \delta(\mathcal{O})$.*[7]

Proof. For $c \in Z_{\mathrm{r}}$, the sequence i_j in Theorem 4.5 is infinite. By the Rigidity Theorem of [AKLS] (or directly from Theorem 4.5), the parapuzzle pieces $\mathcal{V}^{i_j}(c)$ shrink to c. We can now apply Lemma 5.3 with $n = i_j$, which implies the statement (since by Theorem 5.1 combinatorially non-recurrent parameters in $\mathcal{V}^{n+1}$ are almost surely outside $\mathcal{M}$).   □

In order to exploit the previous corollary, we will need the following "Density Points Argument". Let us consider a measurable set $X \subset \mathbb{C}$ such that for almost every $x \in X$ there exists a sequence $X^n(x) \subset \mathbb{C}$ of measurable sets containing x such that $\mathrm{diam}\, X^n(x) \to 0$. Assume that any two $X^n(x)$, $X^m(y)$ are either nested or disjoint. Then $\lim \mathbb{P}(X \mid X^n(x)) = 1$ for almost every $x \in X$. This is a particular case of the standard generalization of the Lebesgue Density Points Theorem (which assumes that the family $\{X^n(x)\}_{x,n}$ satisfies the Besicovitch Covering Property), and can also be seen as a direct consequence of the Martingale Convergence Theorem.

Corollary 5.5. *For almost every $c \in \mathcal{M}$, either f_c has an attracting fixed point, or f_c is renormalizable, or $c \in \mathcal{S}$.*

Proof. It is enough to show that $\mathcal{S}$ has full Lebesgue measure in Z_{r}. For fixed n, the parapuzzle pieces $\mathcal{V}^n(\tilde{c})$ define a partition of Z_{r}. Since the $\mathcal{V}^n(c)$ shrink to c for any $c \in Z_{\mathrm{r}}$, we can apply the Density Points Argument, which implies that $\lim \mathbb{P}(Z_{\mathrm{r}} \setminus \mathcal{S} \mid \mathcal{V}^n(c)) = 1$ for almost every $c \in Z_{\mathrm{r}} \setminus \mathcal{S}$. But by Corollary 5.4, this cannot happen for $c \in Z_{\mathrm{r}}$.   □

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[7] Actually $\delta > 0$ does not depend on c, not even via the Misiurewicz wake: see Remark 4.2.

5.2. Real parameters

Our entire discussion goes through for real parameters as well, without changes. However there are no parameters in $\mathcal{V}^{n+1} \cap \mathbb{R} \setminus \mathcal{M}$, so that we can state the following stronger version of Corollary 5.4:

Corollary 5.6. *There exists $\delta > 0$ such that for $c \in Z_r \cap \mathbb{R}$, there exist parapuzzle pieces $\mathcal{V}^{n+1}(c)$ of arbitrarily small diameter such that $\mathbb{P}(\mathcal{S} \mid \mathcal{V}^{n+1}(c) \cap \mathbb{R}) > \delta$.*

Corollary 5.7. *For almost every $c \in \mathcal{M} \cap \mathbb{R}$, either f_c has an attracting fixed point, or f_c is renormalizable, or $c \in \mathcal{S}$.*

Parameters in $\mathcal{S} \cap \mathbb{R}$ have *exponential decay of geometry*, that is, the ratios λ_n between the lengths of $V^{n+1} \cap \mathbb{R}$ and $V^n \cap \mathbb{R}$ satisfy $\lambda_n < Ce^{-\epsilon n}$ for some $C, \epsilon > 0$. Hence

$$\sum \lambda_n^{1/d} < \infty, \tag{5.4}$$

and by the Martens–Nowicki Criterion [MN] the maps P_c, $c \in \mathcal{S}$, are *stochastic* (that is, they have an absolutely continuous invariant measure).

Remark 5.3. In [BSS1] it is shown that the decay of geometry (that is, $\lambda_n \to 0$) already implies the existence of an absolutely continuous invariant measure.

Corollary 5.8. *The set of non-renormalizable stochastic parameters $c \in \mathcal{M} \cap \mathbb{R}$ has positive Lebesgue measure.*

Corollary 5.9. *Almost every non-renormalizable parameter $c \in \mathcal{M} \cap \mathbb{R}$ is either regular or stochastic.*

Remark 5.4. Corollary 5.8, in the case $d = 2$, was obtained in [J]. The generalization to the higher degree case is well known (see [T, Theorem 2], which follows the approach of [BC]). Our proof is rather different.

Corollary 5.9, in the case $d = 2$, was obtained in [L4] and is new in the higher degree case.

6. Conclusion

By now, we have carried out all the extra work needed for the higher degree case: once we know that the phase-parameter geometry almost surely decays (Corollaries 5.5 and 5.7), the further argument is the same as in the quadratic case. For the reader's convenience, below we will briefly elaborate this statement.

6.1. Real parameters

6.1.1. Collet–Eckmann property (Theorem 1.1). Standard renormalization considerations reduce the analysis of exactly n-times renormalizable parameters with some fixed combinatorics to the analysis of non-renormalizable parameters in a "Multibrot-like family". The analysis of Multibrot-like families is parallel to the one we have done (see [L4] which deals directly, in the case $d = 2$, with Mandelbrot-like families), and one reaches the same theorems, with the difference that all constants may depend on the geometry of the Multibrot-like family under consideration.

Since a renormalizable map is Collet–Eckmann if and only if its renormalization is, Theorem 1.1 follows from the statement that (in a Multibrot-like family) real non-renormalizable parameters are almost surely either regular or Collet–Eckmann. In view of Corollary 5.7, this is reduced to the following result:

Theorem 6.1. *The Collet–Eckmann parameters have full (one-dimensional) Lebesgue measure in $\mathcal{S} \cap \mathbb{R}$.*

This result follows from the statistical argument of [AM1]: as pointed out in Remark 2.1 of that paper, the statistical argument applies for any d to the set of parameters satisfying the following properties:

- $\liminf s_n \geq 1$;
- exponential decay of the real phase geometry (meaning that the ratios of the lengths of the real traces of V^{n+1} and V^n decay exponentially);
- growth of the parameter moduli $\mathrm{mod}(\mathcal{V}^n \setminus \mathcal{V}^{n+1})$.

All these conditions hold for non-renormalizable parameters $c \in \mathcal{S}$ (the exponential decay of geometry follows from the linear growth of the phase moduli $\mathrm{mod}(V^n \setminus V^{n+1})$).

6.1.2. Further statistical properties. The statistical analysis of [AM1] and [AM3] goes far beyond the Collet–Eckmann property, and gives a very detailed description of maps in $\mathcal{S} \cap \mathbb{R}$. As for the Collet–Eckmann property, it can be directly applied to the higher degree case:

Theorem 6.2. *For almost every $c \in \mathbb{R}$ such that f_c is not regular or infinitely renormalizable:*

(1) *The critical point is polynomially recurrent with exponent* 1*:*

$$\limsup \frac{-\ln |f_c^n(0)|}{\ln n} = 1.$$

(2) *The critical orbit is equidistributed with respect to the absolutely continuous invariant measure μ:*

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} \phi(f_\lambda^i(0)) = \int \phi \, d\mu$$

for any continuous function $\phi : I \to \mathbb{R}$.

(3) *The Lyapunov exponent of the critical value,* $\lim \frac{1}{n} \ln |Df^n(f(0))|$, *exists and coincides with the Lyapunov exponent of μ.*

(4) *The Lyapunov exponent of any periodic point p contained in* $\operatorname{supp} \mu$ *is determined (via an explicit formula) by combinatorics (more precisely, by the itineraries of p and of the critical point).*

6.2. Zero area (Theorem 1.3)

Again, by renormalization considerations, Theorem 1.3 reduces to the statement that (in a Multibrot-like family) almost every non-renormalizable parameter is regular. In view of Corollary 5.5, it is thus enough to prove the following statement:

Theorem 6.3. *The set $\mathcal{S}$ has zero area.*

Proof. Fix an arbitrary $c_0 \in \mathcal{S}$ and let $\mathcal{V}^{n+1} = \mathcal{V}^{n+1}(c_0)$. For $c \in \mathcal{V}^{n+1}$, let $g_{n,c}$ denote the first return map to V^n under iteration by f_c, and let $V_*^n(c)$ be the component of $D_n(c) = \operatorname{Dom}(g_{n,c})$ containing the critical value $g_{n,\bar{c}}(0)$.[8] Set

$$\mathcal{V}_*^{n+1} = \{c \in \mathcal{V}^{n+1} : g_{n,c}(0) \in V_*^n(c)\}$$

By the "Density Points Argument" of the previous section, it is sufficient to show that

$$\limsup \mathbb{P}(\mathcal{S} \mid \mathcal{V}_*^{n+1}) < 1. \tag{6.1}$$

For $c \in \mathcal{V}^{n+1}$, let $Z^{n+1}(c)$ be the union of the boundaries of the puzzle pieces that are subordinate to depth v_{n+1} (and hence move holomorphically over $\mathcal{V}^{n+1}$). Persistent puzzle pieces include all components of $D_n(c)$. By Słodkowski's Theorem, the holomorphic motion of Z^{n+1} extends to a holomorphic motion h of the whole complex plane $\mathbb{C}$.

The map $\psi : c \mapsto g_{n,c}(0)$ is a diagonal to the tubing of ∂V_*^n over $\mathcal{V}_*^{n+1}$, so h and ψ give rise to a phase-parameter map $\chi_n : V_*^n \to \mathcal{V}_*^{n+1}$. Since

$$\operatorname{mod}(V^n(c) \setminus V_*^n(c)) \geq \delta n$$

for all $c \in \mathcal{V}^{n+1}$, the first statement of Lemma 2.2 implies that $\operatorname{mod}(\mathcal{V}^{n+1} \setminus \mathcal{V}_*^{n+1}) \to \infty$, so by the λ-lemma, χ_n is γ_n-qc, where $\gamma_n \to 1$.[9]

Given two measurable sets $X \subset Y$ and a bi-measurable injection $\phi : Y \to \mathbb{C}$, we let

$$\mathbb{P}_\phi(X \mid Y) = \mathbb{P}(\phi(X) \mid \phi(Y))$$

be the ϕ-pullback of the conditional probability. Let $\gamma > 1$. Given a Jordan disk V and a measurable set $X \subset D$, let us define the γ-*capacity* $\mathbb{P}_\gamma(X \mid V)$ as follows:

$$\mathbb{P}_\gamma(X \mid V) = \sup \mathbb{P}_\phi(X \mid V),$$

---

[8] In what follows we let $D_n = D_n(c_0)$, and use the similar convention for other objects moving over $\mathcal{V}^{n+1}$.

[9] This kind of rules relating the dynamical and parameter objects are described in [AM2] as the *phase-parameter relation*. For most purposes, one can use these rules axiomatically.

where ϕ ranges over all γ-quasiconformal homeomorphisms $V \to \phi(V) \Subset \mathbb{C}$. Clearly, the γ-capacity is a conformal invariant. Let $\alpha_n = \mathbb{P}_\gamma(D_n \mid V^n)$.

For $n > 0$, the set D_n is "uniformly porous" in V^n in the following sense: There exist $K > 0$, $\mu > 0$ and $\eta > 0$ such that any component W of D_n is contained in the nest of two topological disks, $W \subset W' \subset W'' \subset V^n$, such that:

- $\mathrm{mod}(W'' \setminus W') \geq \mu$;
- W' is a K-quasidisk;
- $\mathbb{P}(D_n \mid W') \leq 1 - \eta$.

To obtain such a nest, take the return map $g_n = g_{n,c} : W \to V^n$, extend it to a branched covering $\hat{g}_n : W'' \to V^{n-1}$ of degree d or 1, and let W' be the pullback by $\hat{g}_n$ of a big intermediate quasidisk U, $V^n \Subset U \Subset V^{n-1}$. Since D_{n-1} is not dense in U (once U is sufficiently large), the Koebe Distortion Theorem implies that D^n has a gap of a definite size in W' (compare Lemma B.3 of [AM2]).

Uniform porosity implies that $\alpha_n < 1$ for $n > 0$ (making use of the Besicovitch Covering Lemma).

Since $g_{n,c} : V^n_*(c) \to V^n(c)$ is a conformal map for $c \in \mathcal{V}^{n+1}$, the connected components of the set

$$\Delta_n(c) = (g_{n,c} \mid V^n_*(c))^{-1}(D_n(c))$$

are puzzle pieces which are subordinate to depth $n + 1$ and hence $\Delta_n(c)$ is respected by the holomorphic motion h. Moreover, for $c \in \mathcal{S} \cap \mathcal{V}^{n+1}$, $g_{n,c}(0) \in \Delta_n(c)$. It follows that $\mathcal{S} \cap \mathcal{V}^{n+1}_*$ is contained in the image of Δ_n under the phase-parameter map $\chi_n : V^n_* \to \mathcal{V}^{n+1}_*$. Since this map is γ-quasiconformal for large n, $\mathbb{P}(\mathcal{S} \mid \mathcal{V}^{n+1}_*) \leq \alpha_n$ by definition of capacity. Thus, to prove (6.1), it is enough to show that

$$\limsup \alpha_n < 1. \tag{6.2}$$

We will obtain this by means of the following simple statistical argument.

Let $O^{n+1} = (g_n \mid V^{n+1})^{-1}(D_n)$. For each connected component W of O^{n+1}, we have

$$\mathrm{mod}(V^{n+1} \setminus W) \geq \frac{1}{d^2}\mathrm{mod}(V^{n-1} \setminus V^n) \geq \delta n. \tag{6.3}$$

Call a component of O^{n+1} *critical* if it contains 0 and *precritical* if its image under g_n contains 0. Let E^{n+1} be the union of critical and precritical components. If $s_n = 0$ (the central return case) then $E^{n+1} = V^{n+2}$; otherwise E^{n+1} consists of $d + 1$ puzzle pieces. In any case, E^{n+1} is the union of at most $d + 1$ puzzle pieces $W_i \subset V^{n+1}$, each satisfying (6.3). It follows that

$$\mathbb{P}_\gamma(E^{n+1} \mid V^{n+1}) \equiv \epsilon_{n+1} \leq e^{-\delta n}. \tag{6.4}$$

Furthermore, if W is a connected component of $O^{n+1} \setminus E^{n+1}$ then $g_n^2 : W \to V^n$ is a conformal map, and $g_n^2(W \cap D_{n+1}) \subset D_n$. It follows that if $\phi : V^{n+1} \to \phi(V^{n+1}) \Subset \mathbb{C}$ is a γ-qc homeomorphism then for any such component W we have $\mathbb{P}_\phi(D_{n+1} \mid W) \leq \alpha_n$ (by the definition of capacity). Hence

$$\mathbb{P}_\phi(D^{n+1} \mid O^{n+1} \setminus E^{n+1}) \leq \alpha_n,$$

so that

$$\mathbb{P}_\phi(D^{n+1} \setminus E^{n+1} \mid V^{n+1}) \leq (1 - \mathbb{P}_\phi(E^{n+1} \mid V^{n+1}))\alpha_n.$$

Thus,

$$\mathbb{P}_\phi(D_{n+1} \mid V^{n+1}) = \mathbb{P}_\phi(E^{n+1} \mid V^{n+1}) + \mathbb{P}_\phi(D_{n+1} \setminus E^{n+1} \mid V^{n+1})$$
$$\leq \mathbb{P}_\phi(E^{n+1} \mid V^{n+1}) + (1 - \mathbb{P}_\phi(E^{n+1} \mid V^{n+1}))\alpha_n$$
$$= \alpha_n + (1 - \alpha_n)\mathbb{P}_\phi(E^{n+1} \mid V^{n+1}).$$

Taking the supremum over all ϕ under consideration, we obtain

$$\alpha_{n+1} \leq \alpha_n + (1 - \alpha_n)\epsilon_{n+1},$$

so

$$\frac{1 - \alpha_{n+1}}{1 - \alpha_n} \geq 1 - \epsilon_{n+1} \geq 1 - e^{-\delta n},$$

which yields (6.2). □

6.3. Porosity of $\mathcal{M}$ (Theorem 1.4)

If c is not combinatorially recurrent, then by Theorem 5.1, c is a Lebesgue density point of the complement of $\mathcal{M}$.

So, assume that c is combinatorially recurrent. By Theorem 4.5, $\bigcap \mathcal{Y}^n(c) = \{c\}$, and by Corollary 5.4 and Theorem 6.3, $\liminf \mathbb{P}(\mathcal{M} \mid \mathcal{Y}^n(c)) < 1$. This is not enough, though, to conclude that c is not a Lebesgue density point of $\mathcal{M}$, since the $\mathcal{Y}^n(c)$ do not in general have a bounded shape (where a set $K \subset \mathbb{C}$ is said to have *C-bounded shape* if it contains a round disk of radius $(1/C) \operatorname{diam}(K)$).[10] However, the following lemma will allow us to replace them with shrinking domains of bounded shape.

Lemma 6.4. *For every $\delta > 0$, there exists $\kappa > 1$ with the following property. Let D be a Jordan disk and let $x \in D$. Then there exists $r > 0$ such that $\mathbb{D}_r(x) \subset D$ and for every Jordan disk $\Delta \subset D$ that intersects both $\mathbb{D}_r(x)$ and $\partial \mathbb{D}_{\kappa r}(x)$, we have $\operatorname{mod}(D \setminus \Delta) < \delta$.*

Proof. Let $\psi : (D, x) \to (\mathbb{D}, 0)$ be the Riemann map, and let r be the maximal radius such that $\psi(\mathbb{D}_r(x)) \subset \mathbb{D}_{1/2}$.

Notice that if $B \subset \mathbb{D}$ is a Jordan disk with $\operatorname{mod}(\mathbb{D} \setminus B) \geq \delta$ and $B \cap \mathbb{D}_{1/2} \neq \emptyset$, then $B \subset \mathbb{D}_R$ where $R = R(\delta) < 1$.[11] By the Koebe Distortion Theorem, $\psi^{-1}(\mathbb{D}_R) \subset \mathbb{D}_{\kappa r}(x)$. If Δ intersects $\mathbb{D}_r(x)$ and $\partial \mathbb{D}_{\kappa r}(x)$, then $\psi(\Delta)$ intersects $\mathbb{D}_{1/2}$ and $\partial \mathbb{D}_R$, so that $\operatorname{mod}(D \setminus \Delta) = \operatorname{mod}(\mathbb{D} \setminus \psi(\Delta)) < \delta$. □

Given $\rho > 0$, there exists n such that $\mathcal{V}^n \subset \mathbb{D}_\rho(c)$, $\operatorname{mod}(\mathcal{V}^n \setminus \mathcal{V}^{n+1}) > \delta$ and $\operatorname{mod}(V^n \setminus V^{n+1}) > \delta$. Then for some $\eta = \eta(\delta)$, we have $\operatorname{mod}(\mathcal{V}^{n+1} \setminus \mathcal{V}^{n+2}(\tilde{c})) > \eta$ and

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[10] One can show that the parapuzzle pieces corresponding to the puzzle pieces in the *enhanced nest* constructed in [KSS] have a bounded shape, but this nest is less convenient for the statistical arguments.

[11] The optimal choice is to take $R \in (1/2, 1)$ such that $\operatorname{mod}(\mathbb{D} \setminus [1/2, R]) = \delta$.

$\mathrm{mod}(V^{n+1}(\tilde{c}) \setminus V^{n+2}(\tilde{c})) > \eta$ for every combinatorially recurrent parameter $\tilde{c} \in \mathcal{V}^{n+1}$. Almost every $\tilde{c} \in \mathcal{V}^{n+1} \cap \mathcal{M}$ is combinatorially recurrent, and by Corollary 5.4 and Theorem 6.3, $\mathbb{P}(\mathcal{M} \,|\, \mathcal{V}^{n+2}(\tilde{c})) < 1 - \epsilon$. By the previous lemma, there exists $r > 0$ such that $\mathbb{D}_r(c) \subset \mathcal{V}^{n+1}$ and any $\mathcal{V}^{n+2}(\tilde{c})$ intersecting $\mathbb{D}_r(c)$ is contained $\mathbb{D}_{\kappa r}(c)$ for some $\kappa > 1$. Let $X \subset \mathbb{D}_{\kappa r}(c)$ be the union of all the $\mathcal{V}^{n+2}(\tilde{c})$ intersecting $\mathbb{D}_r(c)$. Then

$$1 - \mathbb{P}(\mathcal{M} \,|\, \mathbb{D}_{\kappa r}(c)) \geq \kappa^{-2}(1 - \mathbb{P}(\mathcal{M} \,|\, X)) \geq \epsilon \kappa^{-2}.$$

Remark 6.1. Let us indicate how to generalize Theorem 1.4 to the finitely renormalizable case. We cannot just argue via renormalization since it would only prove that a finitely renormalizable parameter is not a density point of a copy of the Multibrot set containing it, and indeed a neighborhood of a satellite renormalizable parameter (with repelling periodic orbits) contains non-renormalizable parameters belonging to infinitely many Misiurewicz limbs.

This can be solved by constructing a different version of the puzzle and parapuzzle, which is designed to be compatible with a fixed renormalization. Namely, one constructs "adapted Yoccoz puzzle pieces of depth 0", where instead of using the external rays landing at the α-fixed point of f_c, one uses the external rays landing at the orbit of the α-fixed point of the renormalization of f_c. Though the combinatorial description is different (see [Sc2], [M3]), the whole geometric and statistical analysis can be carried out to obtain Theorem 1.4 in the more general setting.

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