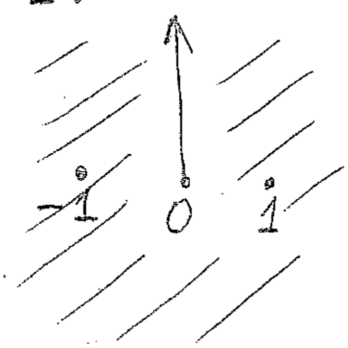
All answers should be justified.

① Which of the following maps are holomorphic?

a) $f(x+iy) = \dfrac{x-iy}{x^2+y^2}$;

b) $f(z) = \bar{z}^2$.

② Write down the branch of the following multi-valued function $f(z)$ in the slit plane $\mathbb{C} \setminus \{iy : y \geq 0\}$ taking the given value at 1:



a) $f(z) = \sqrt{z}$, $f(1) = -1$;

b) $f(z) = \log z$, $f(1) = 2\pi i$;

c) $f(z) = z^i$, $f(1) = 1$.

Calculate $f(-1)$ in each case.

③ Calculate $\displaystyle\int_{\mathcal{C}} \left(z + e^{-\pi z} - \frac{1}{z} \right) dz$

where $\mathcal{C}$ is an arc in the upper half-plane $\{\operatorname{Im} z > 0\}$ connecting i to $2+2i$.

④ (i) Formulate the Cauchy Theorem and state the Cauchy Formula.

(ii) Calculate:

a) $\displaystyle\oint_{|z|=\frac{1}{2}} \frac{z^3 - 2z^2 - z + 3}{z+i}\, dz$;

b) $\displaystyle\oint_{|z|=2} \frac{z^3 - 2z^2 - z + 3}{z+i}\, dz$.

Midterm Solutions

① a) Solution 1. $f(z) = \dfrac{\bar{z}}{|z|^2} = \dfrac{1}{z}$

is holomorphic on $\mathbb{C} \setminus \{0\}$.

(Not entire!)

Solution 2. $u(x,y) = \dfrac{x}{x^2+y^2}$, $v(x,y) = -\dfrac{y}{x^2+y^2}$

Cauchy-Riemann Equations

$$\begin{cases} u_x = \dfrac{y^2-x^2}{(x^2+y^2)^2} = v_y \\ v_x = \dfrac{2xy}{(x^2+y^2)^2} = -u_y \end{cases}$$

are satisfied on $\mathbb{R}^2 \setminus (0,0)$

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b) $\bar{z}^2 = (x-iy)^2 = (x^2-y^2) - i \cdot 2xy$

$u = x^2 - y^2$, $v = -2xy$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ -2y & -2x \end{pmatrix}.$$

CR Equations

$$\begin{cases} 2x = -2x \\ 2y = -2y \end{cases}$$

imply $x=y=0$, so f is differentiable in z only at 0; is nowhere holomorphic.

② For $z=re^{i\theta}$, we have;

$\sqrt{z}=\sqrt{r}\,e^{i\theta/2}$, $\log z=\log r+i\theta$,

$$z^i=e^{i\log z}=e^{-\theta}e^{i\log r}.$$

Branches of these functions are determined by the branches of $\theta=\arg z$ in the slit plane, i.e. $\frac{\pi}{2}+2\pi n<\theta<\frac{\pi}{2}+2\pi(n+1)$,

$$n=0,\pm1,\pm2,\dots$$

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a) Let $n=0$, i.e. $\frac{\pi}{2}<\theta<\frac{5\pi}{2}$.

For this branch $\arg 1=2\pi$, $\arg(-1)=\pi$,

so $\sqrt{1}=e^{i\frac{2\pi}{2}}=e^{i\pi}=-1$, $\sqrt{-1}=e^{i\frac{\pi}{2}}=i$.

b) For the same branch of $\arg z$ we have;

$\log 1=2\pi i$, $\log(-1)=\pi i$.

c) Let $n=-1$, i.e. $-\frac{3\pi}{2}<\theta<\frac{\pi}{2}$.

Then $\arg 1=0$, $\arg(-1)=-\pi$,

and $1^i=1$, $(-1)^i=e^{-(-\pi)}=e^{\pi}$.

3) This function is holomorphic in the upper half-plane, so by the Newton-Leibnitz Formula, the integral is independent of the path and is equal to

$$\left(\frac{z^2}{2}+\frac{e^{-\pi z}}{-\pi}-\log z\right)\Bigg|_{i}^{2+2i}$$ for any branch of log (so, we can take the principal one).

Then we have:

$$\begin{cases} \frac{z^2}{2}\Big|_i^{2+2i}=\frac{(2+2i)^2-i^2}{2}=\frac{8i+1}{2}; \\ e^{-\pi z}\Big|_i^{2+2i}=e^{-\pi(2+2i)}-e^{\pi i}=e^{-2\pi}+1; \\ \log z\Big|_i^{2+2i}=\log 2\sqrt{2}+i\frac{\pi}{4}-i\frac{\pi}{2}=\frac{1}{2}\log 8-i\frac{\pi}{4} \end{cases}$$

Finally,

$$\int=\frac{8i+1}{2}-\frac{e^{-2\pi}+1}{\pi}-\frac{1}{2}\log 8+i\frac{\pi}{4}=$$

$$=\left(\frac{1}{2}-\frac{e^{-2\pi}+1}{\pi}-\frac{1}{2}\log 8\right)+i\left(4+\frac{\pi}{4}\right)$$

(4) a) The function $f(z)=\dfrac{z^3-2z^2-z+3}{z+i}$

is holomorphic on $\mathbb{C}\setminus\{-i\}\supset\{|z|\le\frac{1}{2}\}$.

By the Cauchy Theorem, $\displaystyle\oint_{|z|=\frac{1}{2}} f(z)\,dz=0$.

b) By the Cauchy Formula,

$$\oint_{|\zeta|=2}\frac{p(\zeta)}{\zeta+i}\,d\zeta=2\pi i\,p(-i)=$$

$$=-4\pi+10\pi i \quad (\text{where } p(\zeta)=\zeta^3-2\zeta^2-\zeta+3)$$

Cauchy Thm. Let D be a domain bounded by a simple closed curve $\mathcal{C}$. Let $f: D\to\mathbb{C}$ be a holomorphic function continuous up to the boundary. Then $\displaystyle\oint_{\mathcal{C}} f(\zeta)\,d\zeta=0$.

Cauchy Formula. Under the above circumstances,

$$f(z)=\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

for any $z\in D$.