HW9 p219 #4

Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0 \\ \frac{1}{2} & \text{when } z = 0 \end{cases}$$

is entire.

Solution: $\cos z$ is represented by convergent series:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots$$

for $z \neq 0$ $$(1-\cos z)/z^2 = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \quad (*)$$

So for $z \neq 0$ $f(z)$ is represented by series (*)

By definition of f, $f(0) = \frac{1}{2}$ is equal to the value of the series (*) at 0.

So $f(z) = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \frac{z^6}{8!} \cdots$ is representation of f by a series convergent in the whole plane.

f is entire