Hw 14, p331, #7.

Let $Z = \frac{z-1}{z+1}$. Show that when the principal branch of the square root is used, the composite function

$$w = Z^{1/2} = \left(\frac{z-1}{z+1}\right)^{1/2}$$

maps the z-plane, except for the segment $-1 \le x \le 1$ of the x-axis, onto the right half plane $u > 0$.

Solution

1) First we prove that $Z = \frac{z-1}{z+1}$ maps the segment $-1 \le x \le 1$ of the x-axis onto the segment $X \le 0$ of the X-axis. Clearly it maps real axis to real axis, and $\forall\, x \in [-1, 1]$ $X = \frac{x-1}{x+1} \le 0$.

For $x = 1$ $X = 0$, for $x = -1$ $X = -\infty$. By Intermidiate Value Theorem, X takes on all values in between $-\infty$ and 0. So Z ~~the~~ maps the segment of the x-axis $[-1, 1]$ onto the segment $X \le 0$.

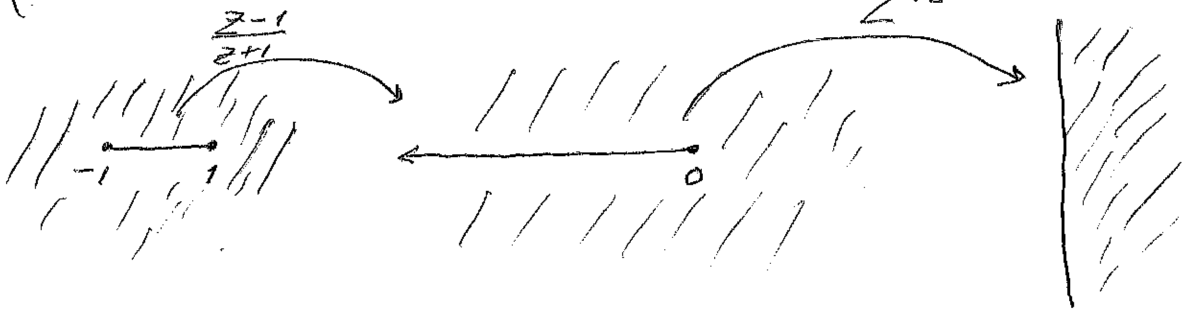
2) Now we prove that $Z = \frac{z-1}{z+1}$ maps

$\{$z-plane $\setminus$ the segment $[-1, 1]$ of the x-axis$\}$

onto the $\{$Z-plane $\setminus$ the segment $X \le 0$ of X-axis$\}$

It follows from the fact (Example 2 Section 102) that $\frac{z-1}{z+1}$ maps the x-axis onto the X-axis and the half planes $y > 0$ and $y < 0$ onto the half planes $Y > 0$ and $Y < 0$ respectively.

3) The principal branch of $Z^{1/2}$ maps
$\{Z\text{-plane} \setminus \text{the ray } X \le 0\}$ onto the right half plane.



So the composite function $\left(\frac{Z-1}{Z+1}\right)^{1/2}$ maps

Z-plane except for the segment $[-1, 1]$ of the x-axis onto the right half plane.