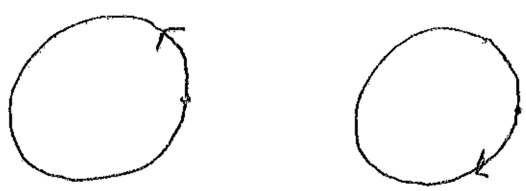
p 306 # 11

Let the circle $|z|=1$ have positive, or couterclockwise, orientation. Determine the orientation of its image under the transformation $w = 1/z$.

Solution

Let $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$.

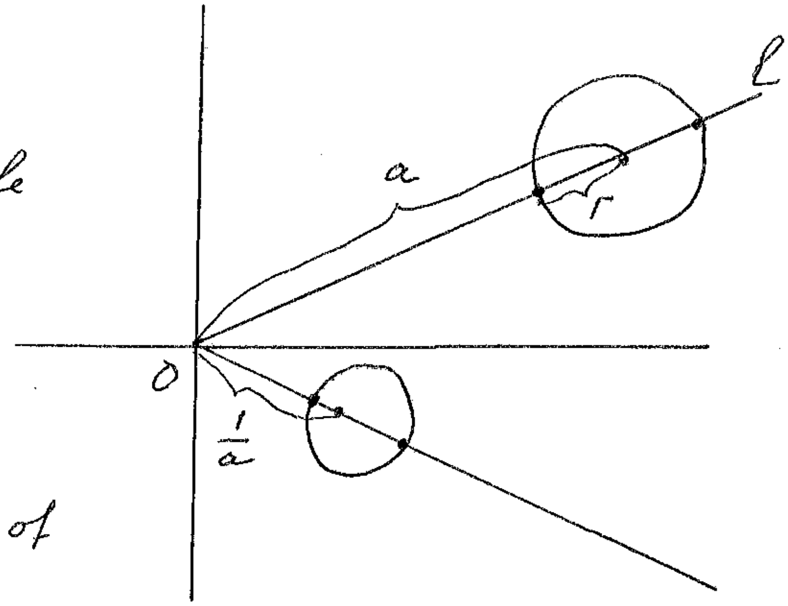
When θ runs from 0 to 2π, z makes a circle in positive, counterclockwise, direction, while $\frac{1}{z}$ makes the same circle in negative, clockwise, direction. So the orientation of the unit circle is changed to the opposite by $w = \frac{1}{z}$.



$z = e^{i\theta}$ $\quad$ $z = e^{-i\theta}$

p 306 #12

Show that when a circle is transformed into a circle under the transformation $w = 1/z$, the center of the original circle is never mapped onto the center of the image circle.



Solution 1

We exclude the case when a circle has center at 0 and its image has center at ∞.

Let a be the distance between the center of the circle and 0, and let r be its radius. Draw a line ℓ through the origin and the center. It intersects the circle at 2 points at distances $a-r$ and $a+r$ from the origin. The images of these points and the center lie on a straight line, image of ℓ, and have distances from the origin $\frac{1}{a+r}$, $\frac{1}{a}$, $\frac{1}{a-r}$ respectively. Since

$$\frac{1}{a} \neq \frac{1}{2}\left(\frac{1}{a+r} + \frac{1}{a-r}\right),$$

the image of the center is not the center of the image circle.

p 306 #12

Solution 2

A circle given by the equation

$$A(x^2+y^2)+Bx+Cy+D=0, \quad B^2+C^2>4AD,$$

has center at $\left(\frac{-B}{2A}, \frac{-C}{2A}\right)$.

The image of this circle under $w=\frac{1}{z}$ is given by the equation

$$D(u^2+v^2)+Bu-Cv+A=0$$ and has center

at $\left(\frac{-B}{2D}, \frac{C}{2D}\right)$.

The image of the center of the original circle

is $\left(\frac{-2AB}{B^2+C^2}, \frac{2AC}{B^2+C^2}\right)$. $\left(\text{Since } (x,y)\mapsto\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)\right)$

If they were equal, ~~then~~

$$\begin{cases} \frac{B}{2D} = \frac{2AB}{B^2+C^2} \\ \frac{C}{2D} = \frac{2AC}{B^2+C^2} \end{cases}$$

then $4AD = B^2+C^2$, contradiction