Problem similar to p247 3b.

Find the order m of the pole and the corresponding reredue B at the singularity $z=0$.

$$\frac{1}{z^2(e^z-1)}$$

Solution

$$\frac{1}{z^2(e^z-1)} = \frac{1}{z^2\left(z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots\right)} = \frac{1}{z^3}\cdot\frac{1}{1+\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots} =$$

$$= \frac{1}{z^3}\,\frac{1}{1-\varphi(z)} =$$

where $\varphi(z) = -\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)$, $|\varphi(z)|<1$ for small $|z|$.

$$= \frac{1}{z^3}\left(1-\varphi(z)+\varphi^2(z)-\dots\right) =$$

$$= \frac{1}{z^3}\left(1-\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}\dots\right)+\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)^2-\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)^3+\dots\right.$$

We need the coefficient at z^{-1} after multiplying by $\frac{1}{z^3}$, i.e the coefficient at z^2 in parantheses.

The terms containing z^2 are: $-\frac{z^2}{3!}$ in $-\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)$ and $\left(\frac{z}{2}\right)^2$ from $\left(\frac{z}{2}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)^2$. The higher terms don't contain z^2.

So $m=3$, $B = -\frac{1}{3!}+\frac{1}{4} = \frac{1}{12}$

p 238, 4a

Evaluate the integral of the function $\frac{z^5}{1-z^3}$ around the circle $|z|=2$ in the positive sense.

Solution

$$\frac{z^5}{1-z^3} = \frac{z^5}{(1-z)(\alpha-z)(\beta-z)}$$ where $1, \alpha, \beta$ are cubic roots of 1. ($\alpha = e^{\frac{2\pi}{3}i}$, $\beta = e^{-\frac{2\pi}{3}i}$)

The f-n has three simple poles whose residues can be calculated. But the way suggested in the book is to use the theorem in section 77 which states that in this case

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\cdot f\left(\frac{1}{z}\right)\right].$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\frac{\left(\frac{1}{z}\right)^5}{1-\left(\frac{1}{z}\right)^3} = \frac{z^3}{z^7(z^3-1)} = \frac{1}{z^4(z^3-1)} = \frac{-1}{z^4(1-z^3)}$$

We look at it as a f-n of z, g(z).

It has a pole of order 4 at 0.

To find its residue at 0 we use geometric progression

$$-\frac{1}{z^4(1-z^3)} = -\frac{1}{z^4}(1+z^3+z^6+\dots)$$

So the residue of g is -1 and

$$\oint f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} g = -2\pi i$$