Homework 1 (due by Thu Sep 17)

1. Cross-ratios and symmetries of the four-punctured spheres

The cross-ratio of four distinct (ordered) points \((z_1, z_2, z_3, z_4) \in \hat{\mathbb{C}}^4\) on the Riemann sphere is defined as

\[
\tau := \left[ z_1 z_2 z_3 z_4 \right] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}.
\]

Warm-up reminder: show that \([z_1 z_2 z_3 z_4] = [\zeta_1 \zeta_2 \zeta_3 \zeta_4]\) if and only if there is a Möbius transformation \(\phi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) such that \(\zeta_i = \phi(z_i)\).

Show that the cross-ratio of permuted points assumes the following six values:

\[
\tau, \quad \frac{1}{\tau}, \quad 1 - \tau, \quad \frac{\tau - 1}{\tau - 1}, \quad \frac{1}{1 - \tau}, \quad \frac{\tau - 1}{\tau}.
\]

Moreover, each of these assumes an arbitrary value in \(\mathbb{C} \setminus \{0, 1\}\).

Hint: bring the points to the normal form \((\tau, 0, 1, \infty)\).

Show that the corresponding six Möbius transformations

\[
\text{id}, \quad \tau \mapsto \frac{1}{\tau}, \quad \ldots, \quad \tau \mapsto \frac{\tau - 1}{\tau}
\]

form the symmetric group \(S_3\) acting on \(\mathbb{C} \setminus \{0, 1\}\). What is the order of each of these transformations? Find the fixed points for each of them. Identify the quotient of \(\mathbb{C} \setminus \{0, 1\}\) modulo this action of \(S_3\).

Given a tuple \((z_1, z_2, z_3, z_4)\), let \(\Gamma\) be the subgroup of the symmetric group \(S_4\) that preserves the cross ratio \(\tau\). Describe this subgroup.

Interpret \(\Gamma\) as the group of symmetries of the Riemann surface \(\mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}\). Describe the “most symmetric” tuples (i.e., the ones whose symmetry group is bigger than a generic symmetry group.)

2. Holomorphic index formula and fixed points of rational functions.

Date: December 10, 2009.
Consider a meromorphic differential
\[ \omega = \sum_{n=N}^{-\infty} a_n z^n \, dz \]

near \( \infty \). What is the order of the pole of \( \omega \) at \( \infty \)? Find \( \text{Res}_\infty \omega \).

Consider a global meromorphic differential \( \omega = f(z) \, dz \) on the Riemann sphere \( \hat{\mathbb{C}} \) (where \( f(z) \) is a rational function). Show that this differential cannot be holomorphic on the whole sphere. Can it have just one simple pole?. Moreover,
\[ \sum_{\text{poles}} \text{Res}_a \omega = 0. \]

A point \( a \) is called a fixed point of \( f \) if \( f(a) = a \). Show that \( f \) has \( d + 1 \) fixed points (appropriately counted with multiplicities), where \( d \) is the degree of \( f \). The set of fixed points is denoted by \( \text{Fix}(f) \).

The multiplier \( \lambda(a) \) of the fixed point is defined as \( f'(a) \) (where the derivative is calculated in an arbitrary local chart near \( a \)). Show that this definition is independent of the local chart.

A fixed point \( a \) is called repelling if \( |\lambda(a)| > 1 \) and it is called parabolic if \( \lambda(a) = 1 \).

Assume that \( f \) does not have parabolic fixed points. Then
\[ \sum_{a \in \text{Fix}(f)} \frac{1}{\lambda(a) - 1} = -1. \]

Hint: consider the differential \( \frac{dz}{f(z) - z} \).

Conclude that any \( f \) with deg \( f > 1 \) has either a parabolic or a repelling fixed point.

Hint: consider the image of the unit disk \( \mathbb{D} \) under the map
\[ \lambda \mapsto \frac{1}{\lambda - 1}. \]

3. Weierstrass \( P \)-function

Let
\[ E_3 = \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{\omega^3}. \]

Show that \( E_3 \) is absolutely convergent.

Conclude that the series
\[ Q(z) = -2 \sum_{\omega \in \mathbb{Z}^2} \frac{1}{(z - \omega)^3} \]
converges uniformly on any domain $D \subseteq \mathbb{C}$ as long as we neglect finitely many terms with poles in $D$. Hence $Q(z)$ is a global meromorphic function on $\mathbb{C}$ with poles of order 3 at the lattice points. Moreover, this function is elliptic (i.e., $Q(z + \omega) = Q(z)$ for any $\omega \in \mathbb{Z}^2$) and odd.

The Weierstrass $\mathcal{P}$-function is now defined as

$$\mathcal{P}(z) = \frac{1}{z^2} + \int_0^z \left( Q(\zeta) + \frac{2}{\zeta^3} \right),$$

$$= \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right), \quad z \in \mathbb{C} \setminus \mathbb{Z}^2$$

(where we take any contour of integration that avoids the lattice pts). Show that it is a well defined global meromorphic function with poles of order 2 at the lattice points. Moreover, it is elliptic and even.

Conclude that $\mathcal{P}$ determines a Galois double branched covering

$$\mathcal{P} : \mathbb{T}^2 \to \hat{\mathbb{C}}.$$ 

What is its Galois group? How many critical points on the torus does $\mathcal{P}$ have? Find these points.

**Homework 2 (due by Thu Oct 1)**

1. **Differential equation for the Weierstrass $\mathcal{P}$-function**

Recall that

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

$$= \frac{1}{z^2} + 3G_2z^2 + 5G_3z^4 + \ldots,$$

where

$$G_k = \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{\omega^{2k}}$$

are the Eisenstein series of weight $k$. Let

$$G_2 = \frac{g_2}{60}, \quad G_3 = \frac{g_3}{140}.$$

Show that the function

$$\phi := (\mathcal{P}')^2 - 4\mathcal{P}^3 + g_2\mathcal{P}(z) + g_3$$

is regular\(^1\) near 0, and $\phi(0) = 0$. Conclude that $\mathcal{P}$ satisfies the following differential equation:

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P}(z) - g_3.$$

\(^1\)a meromorphic function near $a \in \mathbb{C}$ is called regular at $a$ if $a$ is not its pole.
2. Annuli

Uniformize explicitly the strip $\Pi = \{0 < \text{Im} \ z < \pi \}$ by the upper half-plane $\mathbb{H}$ and write down explicitly the \textit{hyperbolic metric} on $\Pi$.

Let $\lambda > 0$. Show that the above uniformization (if selected suitably) conjugates the dilation $\gamma_\lambda : z \mapsto \lambda z$ on $\mathbb{H}$ to the translation $T_h : \zeta \mapsto \zeta + h$ of $\Pi$, with an appropriate $h > 0$ (which one?).

Conclude that the quotient $\mathbb{H}/<\gamma>$ is conformally equivalent to the cylinder $C_h := \Pi/\langle T_h \rangle$.\footnote{Here $<\gamma>$ and $<T>$ stand for the cyclic groups generated by $\gamma$ and $T$ respectively.}

Find the simple closed hyperbolic geodesic in the cylinder $C_h$ (argue that there is only one). What is its hyperbolic length?

Prove that two cylinders $C_h$ with different $h$ are \textit{not} conformally equivalent, so the modulus

$$\text{mod } C_h := \frac{\pi}{h}$$

is a conformal invariant of $C_h$.

Let us also consider round annuli $A \equiv A(r,R) = \{r < |z| < R \}$, $0 < r < R$, and let

$$\text{mod } A := \frac{1}{2\pi} \log \frac{R}{r}.$$  

Find explicitly a conformal isomorphism $\mathbb{C}_h \rightarrow A(r,R)$ (with an appropriate $h$). What does it do to the modulus? Find the simple closed geodesic in $A$.

Conclude that \textit{two annuli} $A(r,R)$ \textit{are conformally equivalent if and only if they have the same modulus}.

3. The punctured disc

Uniformize explicitly the punctured disc $\mathbb{D}^* = \{0 < |z| < 1 \}$ by the upper half-plane $\mathbb{H}$ and write down explicitly the hyperbolic metric on $\mathbb{D}^*$. Draw a picture how $\mathbb{D}^*$ looks like in the hyperbolic metric (this picture is referred to as a “cusp” or a “pseudo-sphere”.)

The discs $\mathbb{D}_r^* = \{0 < |z| < r \}$, $r \in (0,1)$, are called the \textit{horoballs} in $\mathbb{D}^*$. Find the hyperbolic diameter and the hyperbolic area of the horoballs.

Are there closed hyperbolic geodesics in $\mathbb{D}^*$?

Show that $\mathbb{D}^*$ is \textit{not} conformally equivalent to any annulus $A(r,R)$. However, all of them are diffeomorphic. Do you know any other Riemann surface which is diffeomorphic (but not conformally) to $\mathbb{D}^*$? – justify.
4. Hyperelliptic surfaces.

Construct the Riemann surface $S$ of the algebraic function

$$w = \sqrt{P(z)},$$

where $P$ is a polynomial of degree $d$ with simple zeros. Find its genus in two ways: geometrically and by means of the Riemann-Hurwits formula.

Show that $\omega := dz/\sqrt{P(z)}$ is an Abelian differential on $S$. Find its zeros (with multiplicities).

In what follows, assume $d = 5$ and the zeros of $P$ are real. Describe the image of the upper half-plane under the Abelian integral

$$z \mapsto \int^z \omega.$$

Show that $(S, \omega)$ as a flat surface can be obtained by gluing together four polygons. Describe the cone singularities of this surface.

**Homework 3 (due by Thu Oct 15)**

1. **Fubini-Study metric and finite subgroups of PSL(2, $\mathbb{C}$).**

Show explicitly that the projective line $\mathbb{CP}^1$ is biholomorphically equivalent to the Riemann sphere $\hat{\mathbb{C}}$. Under this equivalence the group $\text{PSL}(2, \mathbb{C})$ of projective automorphisms of $\mathbb{CP}^1$ gets naturally identified with the group $\text{M"{o}b}(\hat{\mathbb{C}})$ of Möbius automorphisms of $\hat{\mathbb{C}}$.

Given a Hermitian inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^2$, we can construct the Fubini-Study metric on $\mathbb{CP}^1$ by restricting $\langle \cdot, \cdot \rangle$ to the unit sphere $S^3$ and then projecting it to $\mathbb{CP}^1$. Argue that $\mathbb{CP}^1$ endowed with this metric is isometric to the standard Euclidean sphere. Conclude that $\text{PSU}(2, \mathbb{C})$ (the projective unitary group) is isomorphic to $\text{SO}(3)$ (the special orthogonal group).

Argue that the Fubini-Study metric is independent (up to isometry) of the choice of the inner product.

Select coordinates in $\mathbb{C}^2$ that bring the inner product to the standard form. Write down the Fubini-Study metric explicitly in the corresponding coordinate of $\mathbb{CP}^1 \approx \hat{\mathbb{C}}$.

Prove that any finite subgroup in $\text{PSL}(2, \mathbb{C})$ is conjugate (in $\text{PSL}(2, \mathbb{C})$) to a finite subgroup of $\text{PSU}(2, \mathbb{C}) \approx \text{SO}(3)$. List as many of these groups as you can (up to conjugacy). [Relate Problem 1 from Homework 1 to this classification.]

2. **Congruence subgroup $\Gamma(2)$**.
The congruence subgroup \( \Gamma(2) \) is defined as the subgroup of the modular group \( \Gamma \equiv \text{PSL}(2, \mathbb{Z}) \) consisting of the matrices \( A \) congruent to \( I \) mod 2.

Draw a fundamental domain of \( \Gamma(2) \) and tile it with fundamental domains of \( \Gamma \). Prove that \( \Gamma(2) \) is the free group with two generators. What are those generators? Find the quotient \( G = \Gamma / \Gamma(2) \).

Find the quotient \( \mathbb{H} / \Gamma(2) \) and the action of \( G \) on it. Does it look familiar?

3. Lattés example.

Warm-up: What is the quotient of \( \mathbb{C} \) modulo the the infinite dihedral group generated by \( z \mapsto z + 1 \) and \( z \mapsto -z \)? Can you identify this quotient with a familiar elementary function?

Let us call this function \( \phi \), to keep it secret for a moment. Prove that there is a polynomial \( P_n \) such that \( \phi(nz) = P_n(\phi(z)) \). What are the critical values of the \( P_n \)?

Can you find the \( P_n \) explicitly?

Let \( \Gamma \) be a lattice in \( \mathbb{C} \). Prove that any holomorphic endomorphism \( A : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma \) of the corresponding torus is induced by an affine map \( z \mapsto az + b \) with \( a\Gamma \subset \Gamma \). (Example: \( z \mapsto n\mathbb{Z} \) with \( n \in \mathbb{Z} \) work for any lattice; \( z \mapsto e^{i\pi/4}/\sqrt{2}z \) works for the square lattice.) What is \( \deg A \)?

Let \( z \mapsto az \) be a map as above fixing 0. Prove that there is a rational function \( f = f_{a,\Gamma} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( \mathcal{P}(az) = f(\mathcal{P}(z)) \), where \( \mathcal{P} \) is the Weierstrass \( \mathcal{P} \)-function. What are the critical values of \( f \)?

Find \( f \) explicitly for \( a = e^{i\pi/4}/\sqrt{2} \) on the square lattice.

**Homework 4 (due by Thu Oct 29)**

1. Schwarzian derivative.

Recall that

\[
Sf := \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 = -2\sqrt{f'} \left( \frac{1}{\sqrt{f'}} \right)''
\]

(check the last formula).

Prove that \( S(\phi \circ f) = Sf \) if and only if \( \phi \) is Möbius.

Check the chain rule

\[
S(f \circ g) = (Sf \circ g) \cdot (g')^2 + Sg.
\]

Prove that the Schwarzian equation \( Sf = g \) with a given holomorphic \( g \) extends along any path in the domain of \( g \). (For instance if \( g \) is a polynomial then \( f \) is an entire function.)

Find the general solution of the equation \( Sf \equiv c \).
2. Triangle groups.

Warm-up: Show that for any non-negative $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle with angles $\alpha, \beta, \gamma$.

Take such a triangle $\Delta$ and uniformize it by the upper half-plane $\mathbb{H}$ so that the vertices correspond to 0, 1, $\infty$. Show that this Riemann mapping $\phi : \mathbb{H} \to \Delta$ extends to a mapping $\hat{\phi} : S \to \mathbb{D}$ onto the whole unit disk, where $S$ is a Riemann surface obtained by appropriate gluing of half-planes. Moreover, the natural projection $\pi : S \to \hat{\mathbb{C}}$ is a Galois branched covering. What are the ramification degrees of its branched points? [We allow the ramification degree to be $\infty$: in this case the corresponding point in $\hat{\mathbb{C}}$ is actually omitted from the image of $\pi$.]

Show that the Schwarzian $S\hat{\phi}$ descends to a rational function on $\hat{\mathbb{C}}$. What is the order of its poles?

Assume that the angles of $\Delta$ are $\pi/p, \pi/q, \pi/r$ with integer $p, q, r$ (could be $\infty$). Show that he group $\Gamma$ generated be reflections in the sides of $\Delta$ is discrete, and the translates of $\Delta$ tile the disk $\mathbb{D}$. Moreover, in this case $\hat{\phi} : S \to \mathbb{D}$ is invertible.

Describe the Cayley graph and the algebraic presentation (generators and relations) of $\Gamma$.

3. Hyperbolic metric on plane domains.

Show that on $\mathbb{C} \setminus \{0, 1\}$, the hyperbolic metric near 0 blows up as

$$d\rho \sim -\frac{|dz|}{|z| \log |z|}.$$  

For any domain $U \setminus \hat{\mathbb{C}}$ whose complement contains at least 3 pts, the hyperbolic metric blows up near the boundary, at least as

$$d\rho \geq -\frac{c ds}{|d(z)| \log |d(z)|}, \quad c = c(U) > 0,$$

where $ds$ is the spherical metric and $d(z)$ is the spherical distance from $z$ to $\partial U$.

Homework 5 (due by Thu Nov 12)

1. Holomorphic differentials on hyperelliptic curves.

Let us consider the Riemann surface $S$ of a function $w = \sqrt{P(z)}$, where $P$ is a polynomial with simple roots (assume for definiteness that $\deg P$ is odd). Show that

$$\frac{dz}{w}, \frac{zdz}{w}, \ldots, \frac{z^{g-1}dz}{w}, \quad g = \text{genus of } S,$$
is the basis of holomorphic differentials on $S$.

2. Extremal length.

Consider a flat torus $\mathbb{T}^2 = \mathbb{C}/G$ where $G$ is a lattice generated by $\omega_1$ and $\omega_2$. Let $\Gamma$ be a family of closed curves on $\mathbb{T}^2$ in the homology class $p\omega_1 + q\omega_2$. Show that

$$\mathcal{L}(\Gamma) = \frac{|p\omega_1 + q\omega_2|^2}{\text{area}(\mathbb{T}^2)}.$$ 


The Koebe function is

$$f(z) = \frac{z}{(1-z)^2}.$$ 

Show that it maps univalently $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$. Calculate its Taylor series at $0$. (It is an extremal function for many problems in the geometric function theory.)

3. Fatou and Julia sets.

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, and $f^n = f \circ \cdots \circ f$ be its $n$-fold iterate. A point $z \in \hat{\mathbb{C}}$ belongs to the Fatou set $F(f)$ if it has a neighborhood $U$ such that the family of iterates $\{f^n\}_{n=0}^\infty$ is normal on $U$. Otherwise, $z$ belongs to the Julia set $J(f)$. So, we have a dynamical decomposition $\hat{\mathbb{C}} = F(f) \cup J(f)$. Obviously $F(f)$ is open and $J(f)$ is compact.

Show that both sets are invariant under taking $f$ and $f^{-1}$.

Prove that $J(f)$ is either nowhere dense in $\hat{\mathbb{C}}$ or $J(f) = \hat{\mathbb{C}}$.

Homework 6
(Due by Tue Nov 24 (before the Thanksgiving))

1. Weierstrass points.

A point $p$ on a compact Riemann surface of genus $g$ is called Weierstrass if there is a meromorphic function $\phi \in \mathcal{O}(S)$ with the only pole at $p$ and $\text{ord}_p \phi \leq g$. Show that this is equivalent to existence of a holomorphic differential $\omega$ on $S$ that vanishes at $p$ to order $g$.

Using your knowledge of holomorphic differentials on $S$, identify the Weierstrass points on the hyperelliptic surfaces

$$\{w = \sqrt{P(z)} \}.$$ 

Remark 0.1. It is known that in general there are only finitely many Weierstrass pts.
2. Group structure on elliptic curves.

Consider an elliptic curve
\[ E = \{ w^2 = 4z^3 - g_2z - g_3 \}. \]
We know that it can be uniformized by a standard torus \( \mathbb{C}/\Gamma \) (explicitly by means of the Abelian integral), so it is endowed with a hidden group structure. Let us put 0 of this group at \( \infty \). Show that if a line \( L \) intersects \( E \) at points \( p_1, p_2 \) and \( p_3 \) then \( \sum p_i = 0 \).

3. Field of meromorphic functions on \( E \).

Prove that any meromorphic function on the torus \( \mathbb{C}/\Gamma \) can be expressed as \( R(\mathcal{P}(z), \mathcal{P}'(z)) \), where \( \mathcal{P} \) is the Weierstrass \( \mathcal{P} \)-function and \( R(z, w) \) is a rational function. (Hint: decompose \( \phi \) into even and odd parts.)

Prove that any meromorphic function on the curve \( E \) (0.1) is a restriction of a rational function \( R(z, w) \) to \( E \).

Remark 0.2. This can be generalized to arbitrary smooth algebraic curves.

LAST HOMEWORK, # 7
(due by Tue Dec 8)

1. Limit sets of Fuchsian groups.

Let \( \Gamma \) be a Fuchsian group acting on the unit disk \( \mathbb{D} \). Show that for any two points \( z, \zeta \in \mathbb{D} \),
\[ |\gamma(z) - \gamma(\zeta)| \to 0 \quad \text{as} \quad \gamma \to \infty. \]
Let \( \omega(z) \) be the set of accumulation points of the \( \Gamma \)-orbit of \( z \). Show that \( \omega(z) \subset \mathbb{T} \) and it is independent of \( z \in \mathbb{D} \). This set is called the limit set \( \Lambda_\Gamma \) of \( \Gamma \).

What is \( \Lambda_\Gamma \) for a cyclic group \( \Gamma \)? Prove that \( \Lambda_\Gamma = \mathbb{T} \) for the group of deck transformations of a compact Riemann surface \( S \).

2. Symmetries of hyperbolic Riemann surfaces.

Let \( S \) be a compact hyperbolic Riemann surface, and let \( \text{Aut}(S) \) be its group of conformal automorphisms. Show that \( \text{Aut}(S) \) is compact.

Assume \( A \in \text{Aut}(S) \) is homotopic to \( \text{id} \). Then \( A \) lifts to a Möbius transformation \( \tilde{A} : \mathbb{D} \to \mathbb{D} \) commuting with all deck transformations \( \gamma \in \Gamma \), where \( \Gamma \) is the Fuchsian group that uniformizes \( S \). Moreover, \( \tilde{A} \) fixes the fixed points of the \( \gamma \). Conclude that \( A = \text{id} \).

Derive from the above that \( \text{Aut}(S) \) is finite.
3. Quadratic vs Abelian differentials

Show that a flat structure (with the structure group \(\pm z + c\) and cone singularities) is given by a holomorphic quadratic differential \(q\) if and only if the cone angles are multiples of \(\pi\).

Show that \(q = \omega^2\), where \(\omega\) is an Abelian differential, if and only if the horizontal foliation of \(q\) is orientable.

Show that for any quadratic differential \(q\) on \(S\) there is a double branched covering \(\pi : S' \to S\) such that \(\pi^*q = \omega^2\).

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