Problem Set #9

Solutions

Problem: Let S be a tubular surface about a curve α (see doCarmo, section 2.4, problem 10). Compute the second fundamental form, principal, Gaussian, and mean curvatures of S, in terms of the curvature and torsion of α.

Solution: As we have discussed before, S can be parameterized as

$$\mathbf{x}(s,\theta) = \alpha(s) + r\cos\theta N(s) + r\sin\theta B(s)$$

Computing the derivatives of the parameterization, we get

$$\begin{gathered}\mathbf{x}_s = (1 - r\kappa(s)\cos\theta)T(s) + r\tau(s)\sin\theta N(s) - r\tau(s)\cos\theta B(s)\\ \mathbf{x}_\theta = -r\sin\theta N(s) + r\cos\theta B(s)\end{gathered}$$

and the normal vector to the surface is

$$\nu = \cos\theta N(s) + \sin\theta B(s)$$

The second derivatives of the parameteriztion are

$$\begin{gathered}\mathbf{x}_{ss} = (-r\kappa'(s)\cos\theta - r\kappa(s)\tau(s)\sin\theta)T(s) +\\ (\kappa(s)(1 - r\kappa(s)\cos\theta) + r\tau'(s)\sin\theta - r\tau^2(s)\cos\theta)N(s) +\\ (-r\tau^2(s)\sin\theta - r\tau'(s)\cos\theta)B(s)\\ \mathbf{x}_{s\theta} = r\kappa(s)\sin\theta T(s) + r\tau(s)\cos\theta N(s) + r\tau(s)\sin\theta B(s)\\ \mathbf{x}_{\theta\theta} = -r\cos\theta N(s) - r\sin\theta B(s)\end{gathered}$$

Using this, we can compute the first and second fundamental forms:

$$\begin{gathered}\mathfrak{g} = \begin{pmatrix}(1 - r\kappa(s)\cos\theta)^2 + r^2\tau^2(s) & -r^2\tau(s)\\ -r^2\tau(s) & r^2\end{pmatrix}\\ L_{s\theta} = \begin{pmatrix}\kappa(s)\cos\theta(1 - r\kappa(s)\cos\theta) - r\tau^2(s) & r\tau(s)\\ r\tau(s) & -r\end{pmatrix}\end{gathered}$$

From which we mau compute various curvatures. For example, the Gaussian curvature is

$$K = \frac{\det L_{s\theta}}{\det\mathfrak{g}} = \frac{-\kappa(s)\cos\theta}{r(1 - r\kappa(s)\cos\theta)}$$