Problem Set #2

Solutions

Problem: Prove the following version of the Mean Value Theorem for vector valued functions:

Let $\alpha : [a,b] \to \mathbb{R}^3$ be a regular, smooth curve. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|c-d| < \delta$ then there exists a $\tau \in [c,d]$ for which

$$\big|\alpha(c) - \alpha(d) - \alpha'(\tau)(c-d)\big| < \epsilon\big|c-d\big|$$

Solution: Express the curve in coordinates as $\alpha(t) = (x(t), y(t), z(t))$. For any fixed interval, $[c,d]$, we can apply the standard Mean Value Theorem to the components of α to conclude that there exist real numbers $\tau_x, \tau_y, \tau_z \in [c,d]$ such that

$$\begin{aligned} x(c) - x(d) &= x'(\tau_x)(c-d) \\ y(c) - y(d) &= y'(\tau_y)(c-d) \\ z(c) - z(d) &= z'(\tau_z)(c-d) \end{aligned}$$

or, equivalently, in vetor notation

$$\alpha(c) - \alpha(d) = (x'(\tau_x), y'(\tau_y), z'(\tau_z))(c-d) \tag{1}$$

If we let $\tau = \tau_x$, we no longer have an equality, but by the triangle inequality, we have

$$\begin{aligned} \big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c-d)\big| \le{}& \big|\alpha(c) - \alpha(d) - (x'(\tau_x), y'(\tau_y), z'(\tau_z))(c-d)\big| + \\ & \big|(x'(\tau_x), y'(\tau_y), z'(\tau_z))(c-d) - \alpha'(\tau_x)(c-d)\big| \end{aligned} \tag{2}$$

The first term on the right hand side is zero, by equation (1). Using the fact that $\alpha'(\tau_x) = (x'(\tau_x), y'(\tau_x), z'(\tau_x))$, we can subtract the two vectors in the second term and get

$$\begin{aligned} \big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c-d)\big| &\le \big|(0, y'(\tau_y) - y'(\tau_x), z'(\tau_z) - z'(\tau_x))(c-d)\big| \\ &= \sqrt{(y'(\tau_y) - y'(\tau_x))^2 + (z'(\tau_z) - z'(\tau_x))^2}\,\big|c-d\big| \end{aligned}$$

In order to prove that the term under the radical can be made arbitrarily small, we use the fact that y' and z' are continuous functions, so that $\lim\limits_{\tau_x \to \tau_y} y'(\tau_x) = y'(\tau_y)$. In particular, for every $\epsilon > 0$, there exists a $\delta_y > 0$ such that if $|\tau_y - \tau_x| < \delta_y$, then $|y'(\tau_y) - y'(\tau_x)| < \epsilon/2$. Similarly, using the continuity of z', we can construct a number δ_z sufficiently small to guarantee that $|z'(\tau_z) - z'(\tau_x)| < \epsilon/2$.

Finally, let $\delta = \min\{\delta_y, \delta_z\}$. Since τ_x, τ_y, and τ_z are all contained in the interval $[c,d]$, if $|c-d| < \delta$, then $|\tau_y - \tau_x| < \delta \le \delta_y$ and $|\tau_z - \tau_x| < \delta \le \delta_z$. With this, the previous estimate becomes

$$\big|\alpha(c) - \alpha(d) - \alpha'(\tau_x)(c-d)\big| < \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2}\,\big|c-d\big| < \epsilon\big|c-d\big|$$

as desired.