MAT 141
Problem Set #5
due in recitation on October 7 or 8, 2004

1. Apostol, section 1.7 # 1, 2
2. Apostol, section 1.11 # 1, 2, 6
3. Prove that the union of finitely many rectangles is measurable.
4. We give an inductive definition of a family of sets, $S_n$. $S_0$ is the unit square in the plane; that is, $S_0 = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. To construct $S_{k+1}$ from $S_k$, we do the following. Subdivide the unit square into 9 smaller squares in a tic-tac-toe board pattern. Leave the center square empty and fill each of the remaining squares with a copy of $S_k$, which has been shrunk by a factor of 1/3. The first couple of $S_k$’s are pictured below.

![Diagram of $S_0$, $S_1$, $S_2$, $S_3$, $S_4$]

Let $S = \bigcap S_k$; that is, $S$ consists of those points in the plane that are contained in every $S_k$.
(a) Prove that each $S_k$ is the union of finitely many squares, and hence, measurable.
(b) Derive a formula for the area of $S_k$ and prove that it is correct.
(c) Use the fact that $S$ is contained in each of the $S_k$’s to prove that the infimum of the set
$$\{a(T) \mid T \text{ is measurable and } S \subseteq T\}$$
is zero.
(d) Prove that $S$ is measurable. Compute the area of $S$.
$S_1$ is called the “Sierpinski carpet” and is an example of a fractal, which is short for “fractional dimension”. While we usually think of dimension as
being an integer (a line is one dimensional, a square is two dimensional, and a cube is three dimensional), the dimension of $S$ is $\log_3 8 \approx 1.8928$. 