ON THE REALIZATION OF SYMPLECTIC ALGEBRAS AND RATIONAL HOMOTOPY TYPES BY CLOSED SYMPLECTIC MANIFOLDS

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Abstract. We answer a question of Oprea–Tralle on the realizability of symplectic algebras by symplectic manifolds in the negative in dimensions divisible by four, along with a question of Lupton–Oprea in all even dimensions. This will also imply a negative answer, in all even dimensions six and higher, to another question of Oprea–Tralle on the possibility of algebraic conditions on the rational homotopy minimal model of a closed smooth manifold implying the existence of a symplectic structure on the manifold.

1. Introduction

At the end of his famous two-page paper providing an example of a symplectic non-Kähler compact 4–manifold, Thurston [Th76] posed the following conjecture:

**Conjecture 1.1.** ([Th76]) Every closed 2k–manifold which has an almost complex structure τ and a real cohomology class α such that $\alpha^k \neq 0$ has a symplectic structure realizing τ and α.

Due to foundational results of Taubes and Witten in Seiberg–Witten theory, one can find counterexamples to this conjecture in dimension 4 (the argument to follow has been well-known, see e.g. [Gom01, Example p.49]). Indeed, the oriented connected sum $\#_{i=1}^{2\ell+1} \mathbb{C}P^2$ for any $\ell \geq 1$ contains elements in $H^2$ not squaring to zero and admits an almost complex structure compatible with the orientation, but does not admit a compatible symplectic structure. By a classical result of Wu, one knows that a closed oriented four–manifold $M$ admits an almost complex structure if and only if there is a class $c \in H^2(M;\mathbb{Z})$ such that its reduction mod 2 is the second Stiefel–Whitney class $w_2$ and $\int_M c^2 = 2\chi + 3\sigma$, where $\chi$ is the Euler characteristic and $\sigma$ is the signature. Since $w_2(\#_{i=1}^{2\ell+1} \mathbb{C}P^2) = (1,1,\ldots,1) \in \mathbb{Z}_2^{2\ell+1} \cong H^2(\#_{i=1}^{2\ell+1} \mathbb{C}P^2;\mathbb{Z}_2)$ and $2\chi + 3\sigma = 10\ell + 9$, we see that $c = (3,1,3,1,\ldots,1,3) \in H^2(\#_{i=1}^{2\ell+1} \mathbb{C}P^2;\mathbb{Z})$ satisfies these conditions. Now, if $\#_{i=1}^{2\ell+1} \mathbb{C}P^2$ were to admit a symplectic structure realizing this almost complex structure, then by a theorem of Taubes, since $b_2^+ > 1$ (here $b_2^+$ is the dimension of the positive-definite subspace of the intersection form), it would have a non-vanishing Seiberg–Witten invariant. However, due to Witten, if a manifold is a connected sum of manifolds each with $b_2^+ \geq 1$, then the Seiberg–Witten map is identically zero (see [Ko95, Corollary 4.1(2)]).
In dimensions $\geq 6$, these arguments from Seiberg–Witten theory do not directly apply, and Conjecture 1.1 remains open.

In this note we will address the following variations of this conjecture:

**Conjecture 1.2.** ([OT06, §6.5 Conjecture 3], [HT08], [Tr00]) For every symplectic algebra $H$ over $\mathbb{R}$, there is a closed symplectic manifold $M$ such that $H^*(M; \mathbb{R}) \cong H$.

A *Poincaré duality algebra* (over the field $k = \mathbb{Q}$ or $\mathbb{R}$) of dimension $n$ is a finite-dimensional graded-commutative algebra $H$ over $k$ such that $H^n \cong k$ and the pairing $H^* \otimes H^{n-*} \to k$ given by $\alpha \otimes \beta \mapsto \mu(\alpha\beta)$ is non-degenerate for some (and hence any) choice of non-zero element $\mu \in (H^n)^*$. By a *symplectic algebra* we mean a Poincaré duality algebra of dimension $2k$ for which there exists an element $\alpha \in H^2$ such that $\alpha^k \neq 0$. Hence, for simplicity, the adjectives "Poincaré duality" and "symplectic" will indicate properties of an algebra, not additional structure; $H^*(M; \mathbb{R}) \cong H$ in the above conjecture will mean isomorphism of algebras. In dimensions $n = 4k$, a choice of orientation class $\mu$ lets one consider the signature of the induced pairing on $H^{2k}$. The pairing with respect to $\alpha\mu$ will have the same signature for $a > 0$, and the opposite signature for $a < 0$; thus the signature of a $4k$–dimensional Poincaré duality algebra is well-defined up to sign.

**Question 1.3.** ([LO94, Remark 2.11]) Does a manifold that has rational cohomology algebra a symplectic algebra admit a symplectic structure?

In line with our previous definition, by a manifold we mean a connected orientable closed smooth manifold without a choice of orientation; hence admitting a symplectic (or almost complex) structure means admitting a symplectic form (or almost complex structure) inducing one of the two possible orientations on the manifold. Manifolds with symplectic rational cohomology algebras are also known as *cohomologically symplectic* (or $c$–symplectic) [Tr00].

**Question 1.4.** ([OT06, §6.5 Problem 4], [Tr00]) Are there algebraic conditions on the minimal model $(\mathcal{M}_M, d)$ of a compact manifold $M$ implying the existence of a symplectic structure on $M$?

To answer Conjecture 1.2 in dimensions that are multiples of four, we will use a restriction on the topology of closed almost complex manifolds due to Hirzebruch. For Question 1.3, we will employ simply connected rational homology spheres not admitting spin$^c$ structures in dimensions greater than five. This will immediately imply a negative answer to Question 1.4 when restricted to simply connected manifolds. In the non-simply connected (or more generally, non-nilpotent) case, one must first decide on what is meant by a minimal model in the sense of rational homotopy. However, we observe that any such notion which is invariant under weak homotopy equivalence of rationalizations in the sense of Bousfield–Kan cannot detect the existence of a symplectic form on a given manifold.
2. SOME SYMPLECTIC ALGEBRAS NOT REALIZED BY CLOSED SYMPLECTIC MANIFOLDS

We provide counterexamples to Conjecture 1.2 in dimensions of the form $4k$. Consider for example

$$H = H^* \left( (S^2)^{2k} \# \bigoplus_{i=1}^{j} (S^1 \times S^{4k-1}); \mathbb{R} \right)$$

for odd $j$. Taking $\alpha$ to be the sum of the images of generators of $H^2(S^2; \mathbb{R})$ under the inclusion

$$H^2(S^2; \mathbb{R}) \hookrightarrow H^2((S^2)^{2k}; \mathbb{R}) \hookrightarrow H,$$

we see that $\alpha^{2k} \neq 0$, and so $H$ is a symplectic algebra. Note that the signature $\sigma$ of the realizing oriented manifold

$$(S^2)^{2k} \# \bigoplus_{i=1}^{j} (S^1 \times S^{4k-1})$$

is 0, and so the signature of any oriented manifold $M$ with $H^*(M; \mathbb{R}) \cong H$ is 0, as the signature of a Poincaré duality algebra (with respect to any orientation class) is invariant up to sign under algebra isomorphisms of Poincaré duality algebras. On the other hand, the Euler characteristic satisfies $\chi = 2^{2k} - 2j \equiv 2 \mod 4$ as $j$ is odd. By [Hir87, p.777], a closed almost complex $4k$–manifold with the induced orientation satisfies the congruence $\chi \equiv (-1)^k \sigma \mod 4$, so we conclude that $H$ cannot be realized by an almost complex manifold; in particular it cannot be realized by a symplectic manifold. We emphasize that this conclusion depends only on the algebra $H$, and so we have the following:

**Theorem 2.1.** There are symplectic algebras $H$ over $\mathbb{R}$ in every dimension $4k$, $k \geq 1$, such that there is no closed symplectic manifold $M$ with $H^*(M; \mathbb{R}) \cong H$.

Note that these examples (by taking coefficients in $\mathbb{Q}$ instead of $\mathbb{R}$) provide an answer in the negative to Question 1.3 in dimensions that are multiples of four. Alternatively, we can answer this question negatively in all even dimensions $\geq 6$ as follows: consider the Wu manifold $W = SU(3)/SO(3)$ of dimension 5; this is a simply connected rational homology sphere which does not admit a spin$^c$ structure. We consider the product $S^1 \times W$ and the result of performing surgery on the $S^1$ embedded in this product; this procedure is known as spinning the manifold $W$ [Suc90]. The result of spinning a simply connected rational homology sphere of dimension $n$ is a simply connected [Brow72, Theorem IV.1.5] rational homology sphere of dimension $n + 1$ [Suc90, Lemma 2.1], and the resulting manifold admits a spin$^c$ structure if and only if the original manifold does [AM19, Proposition 2.4]. By iterating this procedure we can produce a simply connected rational homology sphere $M^n$ of any dimension $n \geq 5$ not admitting a spin$^c$ structure, therefore not admitting an almost complex (and in particular a symplectic) structure. For even $n$ we can then take the connected sum $M^n \# \mathbb{CP}^{n/2}$ of this rational homology sphere with $\mathbb{CP}^{n/2}$ to obtain a cohomologically symplectic but not symplectic manifold:
Theorem 2.2. There are cohomologically symplectic manifolds in all dimensions \(2k, k \geq 2\), that do not admit a symplectic structure.

Remark 2.3. Over \(\mathbb{Q}\), one can find symplectic algebras \(H\) that are not realized by any closed smooth manifold. We describe the simplest example in dimension 4; analogues in dimensions that are multiples of 4 are immediate. Take \(H = \mathbb{Q}[x, y]/(y^2 - 2x^2, x^3, y^3, xy)\), where \(x\) and \(y\) are of degree two. If \(H\) were realized as the cohomology of a manifold, then there would be a choice of non-zero \(\mu \in (H^4)^*\) such that the induced non-degenerate symmetric bilinear form \(H^2 \otimes H^2 \to \mathbb{Q}\) given by \(\alpha \otimes \beta \mapsto \mu(\alpha \beta)\) is equivalent to

\[
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}.
\]

Indeed, the fundamental class of the realizing manifold would provide a \(\mu\) such that the induced pairing on \(H^2\) is the rationalization of a unimodular symmetric bilinear form over \(\mathbb{Z}\), and such forms, up to change of rational basis, are representable by diagonal matrices with only \(\pm 1\) on the diagonal [MiHu73, IV.2 Corollary 2.6]. Once a \(\mu\) is chosen, the determinant of the induced bilinear form is well-defined in \(\mathbb{Q}/(\mathbb{Q}^\times)^2\); now note that since the rank of \(H^2\) is even, a different choice of \(\mu\) changes the determinant by a square. Note that the determinant of the pairing on \(H^2\) corresponds to \(2 \in \mathbb{Q}/(\mathbb{Q}^\times)^2\), whereas the determinant of a pairing realized by a manifold would be \(\pm 1 \in \mathbb{Q}/(\mathbb{Q}^\times)^2\).

3. The existence of a symplectic structure cannot be detected from the rational homotopy model

We now address Question 1.4. For any simply connected symplectic manifold \(X\) of dimension at least six, consider the connected sum \(M \# X\) (to form the connected sum we choose any orientation on \(M\) and \(X\)), where \(M\) is a non-spin\(^c\) simply connected rational homology sphere as in the previous section. The collapse map \(M \# X \to X\) is a rational homotopy equivalence, and so the minimal models of these manifolds are isomorphic while only one of them admits a symplectic structure (with respect to some orientation), as \(M \# X\) does not admit a spin\(^c\) structure. Since \(X\) was an arbitrary simply connected symplectic manifold, we conclude that there can be no (non–vacuous) algebraic condition on minimal models of simply connected manifolds which implies the manifold admits a symplectic structure.

In the non-simply connected case, the classical theory for simply connected spaces of finite type due to Sullivan extends immediately to spaces with nilpotent fundamental group which acts nilpotently on the higher homotopy groups, and the algebraic information encoded in the minimal model directly corresponds to geometric information. Bousfield and Kan extended the procedure of rationalizing spaces to all path-connected spaces in two ways [BoKa71]: the \(\mathbb{Q}\)–completion and the fiberwise \(\mathbb{Q}\)–completion, both restricting to the classical rationalization on
nilpotent spaces (see [RWZ19] for an overview). A map \( X \to Y \) induces a weak homotopy equivalence of \( \mathbb{Q} \)-completions if it induces an isomorphism on rational homology [BoKa71], and it induces a weak homotopy equivalence of fiberwise \( \mathbb{Q} \)-completions if it induces an isomorphism on fundamental groups and on rationalized higher homotopy groups (see [RWZ19, Theorem 3]). Substantial progress has been made in algebraically encoding spaces up to these notions of equivalence, extending the classical theory of rational homotopy minimal models; see [GHT00], [BFMT18].

We now observe that for any (not necessarily simply-connected) symplectic manifold \( X \), there is another manifold, not admitting a symplectic structure, which is equivalent to \( X \) under either of the above notions. Consider again the collapse map \( M\#X \to X \), where \( M \) is a non-spin\(^c\) simply connected rational homology sphere; this map induces an isomorphism on rational homology and hence a weak homotopy equivalence of \( \mathbb{Q} \)-completions. The map induces an isomorphism of fundamental groups, and to verify it induces an isomorphism on \( \pi_{\geq 2} \otimes \mathbb{Q} \), we proceed as follows: pick basepoints and consider the induced map \( \widetilde{M\#X} \to \widetilde{X} \) on universal covers. Since \( M \) is simply connected, the space \( \widetilde{M\#X} \) can be visualized as the universal cover of \( X \), with a small disk \( D_i \) around each preimage \( \tilde{b}_i \) of the basepoint of \( X \) (chosen to coincide with the center of the disk at which the connected sum with \( M \) is performed) replaced by \( M\#D_i \). The map on universal covers \( \widetilde{M\#X} \to \widetilde{X} \) is then the collapse map \( M\#D_i \to D_i \) applied at each of these disks, and the identity elsewhere. Now consider the open cover of \( \widetilde{M\#X} \) given by a small neighborhood of \( \bigcup_i (M\#D_i) \) and the complement of \( \bigcup_i (M\#D_i) \), along with the open cover of \( \widetilde{X} \) given by a small neighborhood of \( \bigcup_i D_i \) and the complement of \( \bigcup_i D_i \). Applying the naturality of the Mayer–Vietoris sequence in homology to these open covers, by the five lemma we see that the map \( \widetilde{M\#X} \to \widetilde{X} \) induces an isomorphism on rational homology. Thus, since these spaces are simply-connected, it induces an isomorphism on rational homotopy groups, and from the naturality of the long exact sequence in homotopy for fibrations and the five lemma again we conclude that \( M\#X \to X \) induces an isomorphism on \( \pi_{\geq 2} \otimes \mathbb{Q} \). Therefore the fiberwise \( \mathbb{Q} \)-completions of these spaces are also equivalent. In conclusion, we have:

**Theorem 3.1.** There are no algebraic conditions on the minimal model \((M_M, d)\) of a manifold \( M \) implying the existence of a symplectic structure on \( M \).

Here by a minimal model we mean any object (in particular, the classical minimal models in the case of finite-type nilpotent spaces) which is invariant up to isomorphism under weak homotopy equivalence of rationalizations in either sense of Bousfield–Kan.

We note that the same argument, using non-spin\(^c\) simply connected rational homology spheres, shows that the existence of a (stable) almost complex structure cannot be implied by algebraic conditions on the minimal model:
**Corollary 3.2.** There are no algebraic conditions on the minimal model of a manifold \( M \) implying the existence of a complex structure (or more generally a stable almost complex structure) on \( M \).

**Remark 3.3.** In dimension four, we can find for some rational homotopy types of simply connected manifolds a realizing manifold which does not admit a symplectic structure. The rational homotopy type of a simply connected four--manifold is determined by its second Betti number \( b_2 \) and the absolute value of the signature (with respect to either orientation) [Ter04, Theorem 3]. For \( b_2 \geq 4 \) and any signature \((r, s)\) with \( r, s \geq 2 \), the connected sum of an appropriate number of \( \mathbb{C}P^2 \) and \( \overline{\mathbb{C}P^2} \) will have signature \( \sigma = r - s \) but will not admit a symplectic structure with respect to either orientation by Seiberg–Witten theory as before. For \( r = 1 \) (or \( s = 1 \)), strong results from Seiberg–Witten theory produce non-symplectic four–manifolds for certain values of the signature (e.g. for signature \((1, 9)\) or \((9, 1)\) see [Sza96, Theorem 1.2]). In the case of definite intersection form, the manifold \( \#^r_{i=1} \mathbb{C}P^2 \), for \( r \geq 2 \), will not admit a symplectic structure compatible with either orientation, by combining Hirzebruch’s congruence \( \chi + \sigma \equiv 0 \mod 4 \) for almost complex manifolds and the discussion in the introduction. For \( b_2 \leq 2 \), the only homeomorphism types which admit almost complex structures are the rationally homotopy equivalent \( S^2 \times S^2 \) and \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \), along with \( \mathbb{C}P^2 \); it seems that it is not yet known whether these manifolds admit non-standard smooth structures.

4. Another variation of Thurston’s conjecture

It seems that the following question, another variation of Conjecture 1.1, is still unanswered in all dimensions \( \geq 4 \):

**Question 4.1.** Is there a symplectic algebra which is realized by a closed almost complex manifold but not realized by a closed symplectic manifold?

Currently there are no known topological obstructions to a closed smooth manifold admitting a symplectic structure beyond those of admitting an almost complex structure and having a symplectic cohomology algebra. A possible direction presents itself as it seems that for all known examples of closed symplectic \( 2n \)-manifolds, the Betti numbers \( b_i \) for \( i \leq n \) satisfy the non-decreasing property \( b_0 \leq b_2 \leq b_4 \leq \cdots \) and \( b_1 \leq b_3 \leq \cdots \) [Cho16, Question 1.1]. A proof that this property holds for all closed symplectic manifolds would immediately enable one to provide counterexamples to Conjectures 1.1 and 1.2, along with the above question.

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References


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