A SYMPLECTIC NON-KÄHLER COMPLEX THREEFOLD WITH EVEN b_1,b_3,b_5

Abstract. We give an example of a symplectic 6-manifold (i.e. of real dimension 6) with an integrable complex structure, and with odd Betti numbers being even.

Consider the real Lie algebra given by a basis $\{X_1, X_2, Y_1, Y_2, Z, W\}$ with

$$[X_1, Y_1] = -Z, \quad [X_2, Y_2] = -W,$$

and all other brackets 0. (It is immediate to check that this bracket satisfies the Jacobi identity.) Now take the simply connected Lie group G with this algebra as its Lie algebra, and invoke Malcev to conclude that there is a cocompact discrete subgroup Γ in G. Define a closed, orientable real six-manifold N by setting $N = G/\Gamma$. Its Lie algebra has the structure given above, and so dualizing we obtain a basis for the left-invariant one-forms $\{x_1, x_2, y_1, y_2, z, w\}$ with $dx_1 = dx_2 = dy_1 = dy_2 = 0$, $dz = x_1y_1$, and $dw = x_2y_2$.

Before considering the dual one-forms in more detail, let us observe that we can define an almost complex structure J on N by setting $JX_1 = Y_1, JX_2 = Y_2, JY_1 = -X_1, JY_2 = -X_2, JZ = W, JW = -Z$. Furthermore, this almost complex structure is integrable, since we can immediately see that the Nijenhuis tensor

$$N(A, B) = [A, B] + J[JA, B] + J[A, JB] - [JA, JB]$$

vanishes identically.

Now we consider the dual one-forms. The dga $\Lambda(x_1, x_2, y_1, y_2, z, w)$ with the differential just described is a real minimal model for the nilmanifold N. First, observe that the two-form $\omega = x_1z + x_2w + y_1y_2$ is closed, and $\omega^3 \neq 0$, so we have that N is a symplectic manifold as well. However, this symplectic form is not compatible with the complex structure J (nor is there any such pair of compatible symplectic form and complex structure), as we can see that N is not formal (and hence does not admit a Kähler structure by a result of Deligne, Griffiths, Morgan, and Sullivan). That N is not formal follows from the observation that this six-manifold has four-dimensional first cohomology, spanned by $[x_1], [x_2], [y_1], [y_2]$. If the nilmanifold were formal, it would be a torus, and so its first cohomology would have rank six.

An easy way to see that a given even-dimensional manifold is not Kähler is by invoking the Hodge diamond to conclude that the odd Betti numbers are even. However, let us observe that this approach would not help us conclude that N admits no Kähler structure, as the odd Betti numbers b_1, b_3, b_5 are even. As we already saw, $b_1 = 4$ and so by duality $b_5 = 4$. A direct computation in the minimal model gives us that $H^3(\mathrm{Model}(N), \mathbb{R})$ is spanned by

$$\{[x_1y_1z], [x_1x_2z], [x_1x_2w], [x_1y_2z], [x_1y_2w], [y_1x_2z], [y_1x_2w], [y_1y_2z], [y_1y_2w], [x_2y_2w]\},$$

and so $b_3 = 10$.