THE SPACE OF ALMOST COMPLEX STRUCTURES ON S^6 IS RATIONALLY S^7

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Abstract. Using the algorithm for computing the rational homotopy type of the connected component of a section of a fibration given in [1], we compute the rational homotopy type of the space of almost-complex structures on S^6 to be rationally equivalent to S^7.

1. The space of sections of a fibration

Consider a fibration $F \to E \xrightarrow{p} B$, and suppose F and B are *well-modeled*, meaning that their minimal models (in the sense of rational homotopy theory) correctly compute the rational homotopy groups of the spaces. Suppose further that the fibration is nilpotent, i.e. $\pi_1(B)$ acts nilpotently on the (integral) homotopy groups of F. In the situation we will be considering, B and F will be simply connected and so these conditions of being well-modeled and nilpotent will be automatically satisfied. Denote by Model(F) and Model(B) the minimal models of F and B, with their respective differentials. The minimal model of E is given by $\text{Model}(E) = \text{Model}(F) \otimes \text{Model}(B)$, where the differential is such that restricted to the subalgebra generated by Model(B) it is the differential on Model(B), and on Model(F) it has the form

$$d_E(f) = d_F(f) + \text{ terms in } \text{Model}(B),$$

where d_F is the differential on Model(F).

Now suppose we have a section of this fibration $B \xrightarrow{\sigma} E$, which then induces a map on the level of minimal models $\text{Model}(E) \xrightarrow{\hat{\sigma}} \text{Model}(B)$ such that $\hat{\sigma}\hat{p} = \text{id}_B$, where $\text{Model}(B) \xhookrightarrow{\hat{p}} \text{Model}(E)$ is the inclusion (induced by the projection p). We form a differential graded algebra Γ which models the space of sections in the connected component of σ in the following way:

- Form all possible pairs (f, b^*), where f is an algebra generator for Model(F) and b^* is dual to a basis element of the vector space underlying Model(B). The degree of a pair (f, b^*) is defined to be $\deg(f) - \deg(b)$. If this value is negative, discard this pair. If the degree is zero, replace the pair by the coefficient along b in the element $\hat{\sigma}(f)$. These pairs (f, b^*) form an algebra basis for Γ.
- The differential on Γ is given by

$$d(f, b^*) = (d_E f, b^*) + (-1)^{\deg(f)}(f, \partial b^*).$$

To evaluate $(d_E f, b^*)$, we apply comultiplication on b^* sufficiently many times to obtain an element with wordlength equal to (a summand in) $d_E f$, and pair the elements in the two words piecewise. For example, if $d_E f = f_1 f_2$, and $\Delta(b) = 1 \otimes b_1 + b_2 \otimes b_3 + b_4 \otimes 1$ (where Δ is comultiplication), then

$\Delta(b^*) = 1 \otimes b_1^* + b_2^* \otimes b_3^* + b_4^* \otimes 1$ and we have

$$\begin{aligned}(d_E f, b^*) &= (f_1 f_2, 1 \otimes b_1^* + b_2^* \otimes b_3^* + b_4^* \otimes 1)\\ &= (f_1 f_2, 1 \otimes b_1^*) + (f_1 f_2, b_2^* \otimes b_3^*) + (f_1 f_2, b_4^* \otimes 1)\\ &= (f_1, 1)(f_2, b_1^*) + (f_1, b_2^*)(f_2, b_3^*) + (f_1, b_4^*)(f_2, 1).\end{aligned}$$

In the second term $(f, \partial b^*)$, ∂ denotes the dual to the differential d_B on B. In general, ∂b^* will be a sum $\sum \alpha_i b_i^*$, and so $(f, \partial b^*) = \sum \alpha_i (f, b_i^*)$.

This model is described in [1], and considered in detail in the doctoral thesis of Bora Ferlengez (to appear).

2. Almost complex structures on S^6

The six-sphere is famously only one of two spheres (the other being the two-sphere) that admits an almost complex structure. There are non-vanishing topological obstructions to an almost complex structure on $S^4, S^8, S^{10}, \ldots$ and so no matter the underlying smooth structure, an almost complex structure cannot be placed on the tangent bundle. There is a unique smooth structure on S^2, and a unique complex structure obtained by thinking of S^2 as the complex projective line. The six-sphere also has a unique smooth structure, and an example of an almost complex structure is obtained by interpreting S^6 as the sphere of unit imaginary octonions and letting the endomorphism J of the tangent bundle at a point q be multiplication by q.

We can also do some basic obstruction theory and see that S^6 admits an almost complex structure, and also to observe that this almost complex structure is unique up to homotopy. That is, the space of almost complex structures on S^6 is connected. Indeed, finding an almost complex structure on S^6 is finding a lift of the classifying map $S^6 \to BSO(6)$ of the tangent bundle,

$$\begin{array}{ccccc} & & BU(3) & \longleftarrow & SO(6)/U(3)\\ & \nearrow & \downarrow & & \\ S^6 & \longrightarrow & BSO(6) & & \end{array}$$

The fiber of the map $BU(3) \to BSO(6)$ induced by the inclusion that we want to lift through is given by the quotient $SO(6)/U(3)$, which has the homotopy type of $\mathbb{CP}^3$. So, the obstructions to finding an almost complex structure on S^6 lie in $H^*(S^6, \pi_{*-1}(\mathbb{CP}^3)$, i.e. in $H^6(S^6, \pi_5(\mathbb{CP}^3))$. Since $\pi_5(\mathbb{CP}^3) = \pi_5(S^7) = 0$, there are no obstructions. The obstructions to the uniqueness (up to homotopy) of an almost complex structure lie in $H^*(S^6, \pi_*(\mathbb{CP}^3))$, i.e. in $H^6(S^6, \pi_6(\mathbb{CP}^3))$ which is again zero since $\pi_6(\mathbb{CP}^3) = 0$.

Alternatively, we can consider the problem of finding an almost complex structure as that of finding sections of an $SO(6)/U(3)$ bundle over S^6. This bundle is obtained by replacing the fibers $SO(6)$ of the principal $SO(6)$ bundle associated to the tangent bundle TS^6 by $SO(6)/U(3)$, with the induced action of $SO(6)$ on the quotient $SO(6)/U(3)$.

Now we apply the algorithm from the previous section to compute the rational homotopy type of this space of almost complex structures. Here, the model of the fiber is given by $\text{Model}(\mathbb{CP}^3) = \Lambda(\alpha_2, \beta_7,\ d\alpha = 0, d\beta = \alpha^4)$ and the model of the base by $\text{Model}(S^6) = \Lambda(x_6, y_{11},\ dx = 0, dy = x^2)$. (The subscripts denote the degrees of the generators.) The total space E of the fibration will have model

$$\text{Model}(E) = \Lambda(\alpha, \beta, x, y,\ dx = 0, dy = x^2, d\alpha = 0, d\beta = \alpha^4 + \alpha x).$$

To see that this is in fact the model of the total space, we can observe that the total space is in fact $SO(8)/U(4)$, and compute its model as the cofiber of the map of models induced by the map $BU(4) \to BSO(8)$ on classifying spaces induced by inclusion. To see that the total space is $SO(8)/U(4)$, note that we can obtain $SO(8)/U(4)$ in the following way: $SO(8)/U(4)$ is the space of almost complex structures on $\mathbb{R}^8$ compatible with some fixed orientation. Fix a unit vector e in $\mathbb{R}^8$. To describe an almost complex structure J on $\mathbb{R}^8$, we have to choose what Je will be. This will be a unit vector orthogonal to e, and so will be a point on the unit sphere S^6 in the plane complementary to e. Now on the six-dimensional complement of e and Je we choose an almost complex structure. This gives us a fibration $SO(6)/U(3) \to SO(8)/U(4) \to S^6$, which is in fact the fibration $SO(6)/U(3) \to E \to S^6$ described above.

Note that any space A fitting into a fibration $\mathbb{CP}^3 \to A \to S^6$ has to have minimal model given by $\mathrm{Model}(A) = \Lambda(\alpha, \beta, x, y,\ dx = 0, dy = x^2, d\alpha = 0, d\beta = \alpha^4 + c\alpha x)$ for some constant c. By rescaling the variables x and y we can assume $c = 1$ if it was non-zero to begin with. The case of $c = 0$ corresponds (rationally) to the trivial fibration $\mathbb{CP}^3 \to S^6 \times \mathbb{CP}^3 \to S^6$. So, our total space $SO(8)/U(4)$ is the unique rationally non-trivial fibration over S^6 with fiber $\mathbb{CP}^3 = SO(6)/U(3)$.

Let us proceed with the algorithm for the rational homotopy type of the space of sections Γ of the fibration $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$. For degree reasons, the only algebra generators we have which do not get discarded are

$$(\beta, x^*), (\alpha, 1), (\beta, 1),$$

in degrees 1, 2, and 7 respectively. As for the differential, we have

$$\begin{aligned}
d(\alpha, 1) &= (d\alpha, 1) + (\alpha, \partial 1) \\
&= 0, \\
d(\beta, 1) &= (d\beta, 1) - (\beta, \partial 1) = (\alpha^4 + \alpha x, 1) \\
&= (\alpha^4, 1) + (\alpha x, 1) \\
&= (\alpha, 1)^4. \\
d(\beta, x^*) &= (d\beta, x^*) - (\beta, \partial x^*) = (\alpha^4 + \alpha x, x^*) \\
&= (\alpha^4, 1\otimes1\otimes1\otimes x + 1\otimes1\otimes x\otimes1 + 1\otimes x\otimes1\otimes1 + x\otimes1\otimes1\otimes1) + (\alpha x, 1\otimes x + x\otimes 1) \\
&= 4(\alpha, 1)(\alpha, x^*) + (\alpha, 1)(x, x^*) + (\alpha, x^*)(x, 1) \\
&= (\alpha, 1)
\end{aligned}$$

So, our space of almost complex structures has model

$$\Gamma = \Lambda((\beta, x^*), (\alpha, 1), (\beta, 1),\ d(\beta, x^*) = (\alpha, 1), d(\beta, 1) = (\alpha, 1)^4).$$

Observe that this is not a *minimal* model. The corresponding minimal model is obtained by discarding the superfluous variables (β, x^*) and $(\alpha, 1)$, leaving us with

$$\mathrm{Model}(\text{space of acs on } S^6) = \Lambda((\beta, 1),\ d(\beta, 1) = 0).$$

Recall that the degree of $(\beta, 1)$ is 7, and so this is the minimal model of S^7 as well.

Conjecture.(Sullivan) The space of almost complex structures on S^6 is (integrally) homotopy equivalent to $\mathbb{RP}^7$.

References

[1] Sullivan, D., 1977. Infinitesimal computations in topology. Publications mathématiques de l'IHÉS, 47(1), pp.269-331.