Simply connected manifolds of dimension six or less are formal

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In this note, all models will be over the ground field $\mathbb{R}$. Formality of manifolds descends to subfields, so all conclusions will hold over $\mathbb{Q}$ as well.

A manifold X is said to be *formal* if its minimal model can be constructed by finding the minimal model of its cohomology algebra. That is, we have the following diagram ("zig-zag") of quasi-isomorphisms:

$$\text{minimal model of } X \xrightarrow{\sim} \text{de Rham forms with exterior derivative}$$

$$\text{minimal model of } X \xrightarrow{\sim} H^*(X) \text{ with zero differential}$$

Equivalently, we could require that there exists a map of differential graded algebras from the minimal model of X to the cohomology algebra that induces the identity on cohomology.

We follow the proof given in [1] Fernández-Muñoz "Formality of Donaldson Submanifolds" that simply connected manifolds of dimensions two through six are formal. First a characterization of formality is given, upon which a weaker notion of formality is introduced (*s-formality*) by relaxing the conditions of this characterization.

The mentioned characterization of formality is the following, used in Deligne-Griffiths-Morgan-Sullivan (wherein they prove that all closed Kähler manifolds are formal):

Proposition. Denote the underlying algebra of the minimal model of a manifold X by $M = \Lambda V$, where V is a graded vector space of generators of the minimal model. Let V_i denote the i-th grading in V, and let C_i denote the subspace of closed elements in V_i. The manifold X is formal if and only if we can choose complements N_i to the C_i (so that $V_i = C_i \oplus N_i$) such that any closed element in the ideal generated by $\oplus_i N_i$ in ΛV is exact.

Proof. Suppose X is formal, i.e. we have a map of dga's $f : M \to H^*(X)$ inducing the identity on cohomology. Since the differential in the minimal model is decomposable (by definition of the model), the inclusion of the closed elements in the space of degree i generators V_i into the cohomology algebra is injective (because no sums can be made exact, by minimality). Hence f is injective on C_i as well. Setting N_i to be the kernel of f within V_i, we obtain $V_i = C_i \oplus N_i$. If some $a \in \text{ideal}(\oplus_i N_i)$ is closed, then $f(a) = 0$ and thus $0 = f^*(a) = [a]$, so a is exact.

Conversely, suppose we have complements N_i to the C_i as described above. On each V_i, define a map $f : V_i \to H^*(X)$ by projecting to C_i and taking its cohomology class. Extend this map by multiplicativity to all of ΛV. We show that this map is a map of dga's, i.e. $df = fd$. Since the image of f lies in the cohomology algebra, which has trivial differential, we have $df = 0$. We now show that $f(d(a)) = 0$ for all $a \in \Lambda V$. Write $da = c + n$, where c consists of elements purely in the spaces C_i, and n lies in the ideal generated by $\oplus_i N_i$. Observe $dn = dda - dc = 0$ (since c is closed), so by assumption

n is exact, $n = dn'$. Then $c = d(a - n')$, so $[c] = 0$ and thus $f(d(a)) = [c] = 0$. Note that f induces the identity on cohomology.

Now we define the notion of s-formality, where s is any non-negative integer. For a minimal dga $M = \Lambda V$, we say it is s-formal if, for $i \leq s$ we can choose complements N_i to the space of closed generators C_i in V_i such that any closed element in the ideal $\oplus_{i \leq s} N_i$ within $\Lambda V^{\leq s}$ is exact in ΛV. We say a space is s-formal if its minimal model is s-formal (this property is dga-homotopy invariant).

Note first that s-formality imples $s-1$-formality, and that formality implies s-formality for all s. The utility of this new notion becomes apparent once it is proved that a manifold of dimension $2n$ or $2n-1$ is formal if and only if it is $n-1$-formal. To assist us in proving this result, we need two preliminary observations.

Lemma. An n-manifold X is formal if and only if it is n-formal.

Proof. As is evident from the definition of n-formality and the preceding proposition, formality implies n-formality. Conversely, suppose X has an n-formal minimal model ΛV. We will show that the conditions of the proposition hold, i.e. we will find subspaces $N_i \subset V_i$ complementary to the subspaces of closed generators C_i such that any closed element in the ideal generated by $\oplus_i N_i$ is exact. For $i \leq n$ take the N_i to be the ones provided by virtue of n-formality. For $i > n$ set $N_i = V_i$. Now consider a closed element a in the ideal generated by $\oplus_i N_i$ in all of ΛV. If the degree of a is no more than n, then a is in fact in the ideal generated by $\oplus_{i \leq n}$ in $\Lambda V^{\leq n}$ for degree reasons, and so by assumption it is exact. If the degree of a is greater than n, then we know that it must be exact since X has no cohomology above its dimension.

Let us omit the proof of the following lemma. Details can be found in [1] (along with everything else discussed here).

Lemma. The product of two manifolds, $X \times Y$, is s-formal (for any s) if and only if both X and Y are s-formal.

Now we come to the result that will let us easily conclude formality of manifolds with some connectivity assumptions.

Theorem. An oriented manifold X of dimension $2n-1$ or $2n$ is formal if and only if it is $n-1$-formal.

Proof. The orientability hypothesis is necessary since we will use that top cohomology is one-dimensional, and we will use Poincaré duality.

Dimension $2n$. Suppose we have a minimal model ΛV of X which is $n-1+r$-formal, for some $r \geq 0$. (The case $r = 0$ is the initial assumption.) We construct an $n+r$-formal minimal model of X. Consider the generators $\{x_1, x_2, \ldots\}$ of V^{n+r} (where V^{n+r} is the vector space of pure degree $n+r$ elements in V), ordered so that if dx_i does not contain $x_{i+1}, x_{i+2}, \ldots$. We will go one by one through these generators, defining spaces $\hat{V}_i$ which decompose into $\hat{V}_i = \hat{C}_i \oplus \hat{N}_i$ such that any closed element in the ideal generated by $\hat{N}_i$ in $\Lambda \hat{V}_i$ will be exact in ΛV. Each of $\hat{V}_i, \hat{C}_i, \hat{N}_i$ will be contained in $\hat{V}_{i+1}, \hat{C}_{i+1}, \hat{N}_{i+1}$ respectively, and so this construction will give us the desired conclusion of $n+r$-formality when we go through all of the (finitely many) generators $x_1, x_2, \ldots$ of V^{n+r}.

If x_i is a generator of V^{n+r} such that there exists a $y \in \Lambda V^{\leq n-1+r}$ with $dy = dx_i$ (note that this happens, in particular, if x_i is closed), then define $\hat{x}_i = x_i - y_i$, observe $d\hat{x}_i = 0$, and set

$\hat{V_i} = V^{\le n-1+r} \oplus \mathrm{span}(\hat{x_1}, \dots, \hat{x_{i-1}})$. Start the procedure announced in the previous paragraph by reordering the generators $x_1, x_2, \dots$ so that all generators such that there exists such a y are at the beginning of the sequence (their order among themselves does not matter), and define $\hat{V_i}$ as above. Set $\hat{C_i} = \hat{C}_{i-1} \oplus \mathrm{span}(\hat{x_i})$ and $\hat{N_i} = \hat{N}_{i-1}$, with initial data $\hat{V_0} = V^{\le n-1+r}, \hat{C_0} = C^{\le n-1+r}, \hat{N_0} = N^{\le n-1+r}$. The elements in $\hat{C_i}$ are certainly all closed, by construction, so it just remains to check that any closed element η in $\hat{N_i} \cdot \Lambda\hat{V_i}$ is exact in ΛV. Observe $\hat{N_i} \cdot \Lambda\hat{V_i} = \hat{N}_{i-1} \cdot \Lambda(\hat{V}_{i-1} \oplus \mathrm{span}(\hat{x_i}))$ and so

$$\eta = \eta_0 + \eta_1\hat{x_i} + \eta_2\hat{x_i}^2 + \cdots + \eta_k\hat{x_i}^k$$

with $\eta_i \in \hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$. Since $d\eta = 0$ and $d\hat{x_i} = 0$, we conclude that $d\eta_0 + d\eta_1 \cdot \hat{x_i} + \cdots + d\eta_k \cdot \hat{x_i}^k = 0$ and so, since ΛV is free, every $d\eta_j$ is zero, and so each η_j is exact by induction on i. Therefore η is exact as well.

Now consider the remaining generators, i.e. those for which we cannot find such a y as above. In the steps we do for these generators, we will have $\hat{C_i} = \hat{C}_{i-1}$. To define $\hat{N_i}$, we will add the span of a modified x_i to $\hat{N}_{i-1}$. We discuss this modification of x_i now. Let us pretend that no modification of x_i is necessary, and let us set $N_i = \hat{N}_{i-1} \oplus \mathrm{span}(x_i)$. Take a closed element in the ideal of N_i in $\Lambda(\hat{V_i} \oplus \mathrm{span}(x_i))$ and let us see what we can say about it (ideally, it would be closed).

So, take a closed element $\eta \in (\hat{N}_{i-1} \oplus \mathrm{span}(x_i)) \cdot \Lambda(\hat{V_i} \oplus \mathrm{span}(x_i))$. We can write

$$\eta = \eta_0 + \eta_1 x_i + \cdots \eta_k x_i^k,$$

with $\eta_j \in \hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$. If $k = 0$, that is $\eta = \eta_0$, then by induction on i this is exact. If $k \ge 3$, then the degree of η is strictly greater than $2n$ since the degree of x_i is at least n, and so η is exact. If $k = 2$ and the degree of x_i is $n + 1$ or greater, then we have the same conclusion. If $k = 2$ and x_i has degree n, we can write $\eta = \eta_0 + \eta_1 x_i + c x_i^2$, where c is a real number. Since $d\eta = 0$, we have $(d\eta_0 + \eta_1 dx_i) + (d\eta_1 + 2c dx_i)x_i = 0$, so again by freeness of the model $d(\eta_1 + 2cx_i) = 0$. Since η_1 is obtained from the previous stages, we can write $\eta_1 = a + b$, with $a \in \Lambda V^{\le n-1}$ and $b \in \mathrm{span}(x_1, \dots, x_{i-1})$. Since $d(b - 2cx_i) = da$, we conclude that $b - 2cx_i$ should be a linear combination of those generators we considered first (two paragraphs ago). However, by construction this is not the case, since we are considering the "remaining generators" x_i (introduced in the previous paragraph).

Now it remains to consider the case where our closed element η is of the form $\eta = \eta_0 + \eta_1 x_i$. Note that we can decompose η into a sum of elements of pure degree, so we can assume η is of some pure degree. If the degree of η is $2n + 1$ or more, then η is exact since X is a $2n$-manifold. Let us now focus on the case of η with degree $2n$. Observe that $d\eta = (d\eta_0 + \eta_1 dx_i) + (d\eta_1)x_i$, so η_1 is closed, and since the degree of x_i is $n + r$, the degree of η_1 is $n - r$. We will show that we can modify x_i slightly so that every expression of the form $\eta_0 + \eta_1 x_i$ with $\eta_0 \in \hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$ and $\eta_1 \in \Lambda\hat{V}_{i-1}$ (as we have) is exact if η_1 is closed. Take a basis $\{\hat{\eta}_j\}$ of the vector space of elements of degree $n - r$ in $\Lambda\hat{V}_{i-1}$ such that there exists an element $r_j \in \hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$ of degree $2n$ such that $r_j + \hat{\eta}_j x_i$ is closed. The element $r_j + \hat{\eta}_j x_i$ is to mimic the element $\eta_0 + \eta_1 x_i$, with r_j acting as η_0 and $\hat{\eta}_j$ acting as η_1. We are assuming this sum is of pure degree $2n$. Note that each of these elements is of top degree. Therefore, denoting by ω a volume form for X (here we finally use orientability of X), there is a real number λ_j and a $2n - 1$-form ξ_j such that $r_j + \hat{\eta}_j x_i = \lambda_j\omega + d\xi_j$ for each of the basis elements $\hat{\eta}_j$. This number λ_j does not depend on the choice of r_j or ξ_j for a given $\hat{\eta}_j$. Indeed, if we have an alternate equation $r'_j + \hat{\eta}_j x_i = \lambda'_j + d\xi'_j$ alongside $r_j + \hat{\eta}_j x_i = \lambda_j + d\xi_j$, then

$$r_j - r'_j = (\lambda_j - \lambda'_j)\omega + d(\xi_j - \xi'_j)$$

is a closed element in $\hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$, and so it is exact by induction, so $\lambda_j = \lambda'_j$.

Now we show that with an appropriate modification of x_i, we can make all the λ_j equal to 0, and so our desired conclusion of $\eta_0 + \eta_1 x_i$ being exact would follow. First let us consider those basis elements $\hat{\eta}_j$ which are exact. Then $\hat{\eta}_j = d\alpha$ for some α of degree $n - r - 1$. Writing α as a sum of a closed element and an element in the ideal generated by $N^{\leq n-r-1}$, we see that we can forget about the closed summand and just take $\alpha \in N^{\leq n-r-1} \cdot \Lambda V^{\leq n-r-1}$. Then

$$r_j + \hat{\eta}_j x_i = r_j + (d\alpha)x_i = r_j + d(\alpha x_i) \pm \alpha dx_i,$$

so $r_j \pm \alpha dx_i$ is closed and hence exact by induction since it lives in $\hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$. Therefore $r_j + \hat{\eta}_j x_i$ is exact as well, and so $\lambda_j = 0$.

Let us now consider those basis elements $\hat{\eta}_j$ which are not exact. By scaling and adding up their Poincaré dual elements (due to orientability again), we find a closed element y of degree $n+r$ such that $[\hat{\eta}_j][y] = \lambda_j[\omega]$ for all considered j. Since y is closed, it must be a combination of elements in $\Lambda V^{\leq n-r-1}$ and generators x_l of V^{n+r} of the first kind considered. So, y comes from one of the previous stages (in relation to our current x_i). Now we do the announced modification of x_i by setting

$$\hat{x_i} = x_i - y.$$

Note that this modification does not change the conclusion of exactness in the cases where η was of the form $\eta = \eta_0$ or $\eta = \eta_0 + \eta_1 x_i + \cdots + \eta_k x_i^k$ since the actual form of x_i did not matter. Also, the conclusion of $\lambda_j = 0$ for the exact basis elements $\hat{\eta}_j$ still holds. Now observe that, for the non-exact basis elements $\hat{\eta}_j$, we have

$$\begin{aligned}[r_j + \hat{\eta}_j \hat{x_i}] &= [r_j] + [\hat{\eta}_j][x_i] - [\hat{\eta}_j][y] \\ &= \lambda_j[\omega] - \lambda_j[\omega] = 0.\end{aligned}$$

So, finally, we have the desired conclusion that $\eta_0 + \eta_1\hat{x}_i$ is exact (albeit along the way we have modified x_i; however, the end goal is to construct an $n+r$-formal minimal model, so modification of the originally chosen basis of V^{n+r} is no issue).

Let us set $\hat{N}_i = \hat{N}_{i-1} \oplus \operatorname{span}(\hat{x_i})$. It only remains to check that $\eta = \eta_0 + \eta_1\hat{x_i}$ as considered earlier, but now of degree at most $2n-1$, is exact. Note that $[\eta_0 + \eta_1\hat{x_i}]$ is in the upper half of cohomology, since its degree is at least that of $\hat{x_i}$, which is $n+r$. Taking test elements $[\gamma]$ in the corresponding dual class (which is in the lower half), we have

$$[\gamma][\eta_0 + \eta_1\hat{x_i}] = [(\gamma\eta_0) + (\gamma\eta_1)\hat{x_i}].$$

Since $\gamma\eta_0$ is in $\hat{N}_{i-1} \cdot \Lambda\hat{V}_{i-1}$ and $\gamma\eta_1$ is closed (since both γ and η_1 are), we conclude that $[\gamma][\eta_0 + \eta_1\hat{x_i}] = 0$ (the modification of x_i to $\hat{x_i}$ was done for this very reason, to make such elements exact). Since the pairing on cohomology is nondegenerate, this can only mean that $\eta_0 + \eta_1\hat{x_i}$ itself if exact. Now we have finally gone through all the possible cases, and constructed an $n+r$-formal minimal model from an $n+r-1$-formal one. By induction we conclude that we can construct an n-formal model, and thus (by an earlier lemma) X is formal.

Dimension $2n-1$. If X is of dimension $2n-1$ and has an $n-1$-formal minimal model, then consider the $2n$-manifold $X \times S^1$. The circle is formal, and so it is $2n-1$-formal, so $X \times S^1$ is as well. By the preceding case, we have that $X \times S^1$ is thus formal, and so the factor X is formal as well. □

Now let us consider oriented manifolds of low dimension. Observe that a connected manifold is 0-connected. Therefore 1-manifolds and 2-manifolds are formal. However, there was no need for such machinery. Circles are formal (their minimal model is $\Lambda(x_1)$) and oriented surfaces are connect sums of tori. A torus is formal (as a product of circles), and formality is preserved under connect sum.

Three-manifolds need not be formal. The quotient of the real Heisenberg group by its integer lattice is a non-formal closed manifold of dimension three. Let us consider its details some other time. A simply connected three-manifold, however, is (integrally) homotopy equivalent to a sphere, and hence formal.

The integral homotopy type of a simply connected four-manifold is determined by its intersection form, which is equivalent information to its cohomology ring. Therefore simply connected four-manifolds are formal.

The theorem lets us conclude more: simply connected manifolds of dimensions 5 and 6 are also formal. Indeed, it suffices to show that they are 2-formal. Consider the minimal model ΛV of such a manifold. Then $V_1 = 0$. Note that every element in degree two in closed. Indeed, applying the differential to any degree two element gives a decomposable degree three element, but there are no such elements (except 0) since $V_1 = 0$. Therefore taking $C_1 = N_1 = 0$ and $C_2 = V_2, N_2 = 0$, we see that ΛV is 2-formal and thus formal.

The preceding argument for formality has the following slight generalization.

Theorem. An l-connected ($l \geq 1$) manifold X is formal if $\dim X \leq 4l + 2$.

Proof. We show that X is $2l$-formal. The assumption of l-connected gives us that there is a minimal model of X with no elements of degree l or less. Since the differential is decomposable, this means that the lowest degree on which the differential can act non-trivially is $2l + 1$ (since the differential of such an element can be a sum of products of two elements of degree $l + 1$). Therefore we can take $C_i = V_i$ and $N_i = 0$ for $i \leq 2l$, and we have $2l$-formality.

We can extend the previous theorem to $l = 0$ (i.e., connected manifolds) if we add the assumption of orientability. This additional assumption is necessary. Indeed, $\mathbb{RP}^2$ has the real cohomology ring of a point, though it has non-trivial homotopy groups.

References.

[1] Fernández, Marisa, and Vicente Muñoz. "Formality of Donaldson submanifolds." Mathematische Zeitschrift 250.1 (2005): 149-175.

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[3] Deligne, P., Griffiths, P., Morgan, J., & Sullivan, D. (1975). Real homotopy theory of Kähler manifolds. Inventiones mathematicae, 29(3), 245-274.